

UNIT-I

LINEAR PROGRAMMING PROBLEM

INTRODUCTION TO OPERATIONS RESEARCH

INTRODUCTION

Making a decision is becoming more and more complex. One has to take certain decisions for himself and for others. Managers are evaluated primarily on the results of their decisions and successful managers use quantitative approaches to decision making.

These approaches are also known as:

Operations Research [OR]

Management Science [MS]

Decision Science [DS]

Quantitative Techniques for Business Decision [QTBD]

Operation Research is an application of scientific approach to solve management problems in order to help managers to make better decisions. An effective decision depends on many factors, which may be economic, social and political.

Example:

Starting of a new factory at a place would depend on economic factors such as:

- a. Construction costs
- b. Labor costs
- c. Availability of raw material
- d. Transportation costs
- e. Taxes
- f. Energy costs
- g. Pollution control costs etc.,

Qualitative and Quantitative Analysis

While solving a real life problem the decision maker must examine a problem from both qualitative as well as quantitative perspective, so that data so obtained should be analyzed in both perspectives to suggest solution of problem

- ***Qualitative analysis***

It is primarily based on the judgment and experiences of the managers and it includes the manager's feelings regarding the problem. Qualitative analysis is emphasized in the situations where the managers have experiences with similar problems or if the problem is relatively simple.

- ***Quantitative Analysis***

It is recommended if the problem is quite complex and the manager is not having enough experience with similar problems.

Quantitative analyses are primarily mathematical and statistical techniques are used in managerial decision making. Quantitative approach to problem solving requires collection of large amount of data and development of mathematical models. Quantitative techniques are used to support

management to make decisions under complex and uncertain situation in order to achieve the objective and goal the organization.

Example:

Consider a problem of an investor seeking advice for investments in three alternatives; stock market, real estate and bank deposit. To suggest an acceptable solution, we need to consider certain quantitative factors to be examined in the light of the problem.

ORIGIN AND GROWTH/EVOLUTION/HISTORY OF OR

However, a particular model and technique of OR was used as early as in World War-I

- In the year 1914-1915, Thomas Edison used a tactical game board to minimize shipping losses from enemy submarines instead of risking ships in actual war conditions
- About the same time, A.K Erlang developed experiments to study the fluctuations in demand for telephone facilities using automatic dialing equipment, later on such experiments were used for the development of waiting line theory.

Stage: 1: During World War-II

It is generally agreed that OR came into existence during WW-II when there was a critical need manage scarce resources in military operations. Since the war involved strategic and tactical problems which were greatly complicated and provided adequate solutions.

Therefore, groups of individuals who collectively were considered as specialists in Mathematics, Economics, Statistical and Probability theory, Engineering, Behavioral and Physical science were formed as special units within the armed forces to deal with strategic and tactical problems of various military operations.

Such groups were first formed by the British air force and later, the American armed forces formed similar groups

- At very first Britain developed radar operational research unit and was assigned to analyze the problem of radar at gun sites under the guidance of Prof. PMS Blackett which also called as “Air Battle of Britain”
- Later, United States, Canada and France started the use of OR techniques with such team of scientists.
- At the end of WW-II, the scientists of this team moved to different sectors, such as transportation, health, Education.

Stage: 2: After World War-II

After the war ended, OR approaches applied to civilian problems, business, industry, research and development etc.

There are various important factors behind the rapid development in the use of operations research approach.

- After WW-II, The economic and industrial boom resulted in continuous mechanization, automation, decentralization of operations and division of management functions and resulted in complex managerial problems. Therefore applications of OR to managerial decision-making became popular.

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- In the year 1947, George B Dantzig, developed the concept of Linear Programming to allocate scarce resources by a method known as simplex method.
- At the end of 1950, many other techniques of OR developed such as SQC, Dynamic programming, queuing theory and inventory theory
- A year later, project scheduling techniques: CPM and PERT were developed as efficient tools for scheduling and monitoring lengthy, complex and expensive projects of that time.
- In 1948, an OR club was formed in England which later changed its name to the OR society of U.K. its journal, OR quarterly first appeared in 1950
- In 1952, the OR society of America (ORSA) was founded and its journal operations research was first published in 1953.
- In 1953, The Institute of management sciences [TIMS] was founded as an international society to identify, extend and unify scientific knowledge pertaining to management. Its journal, management science, first appeared in 1954.

Stage: 3: In India

In India operations research originated during the last phase of WW-II. Initially, an OR Centre was formed at the Air Head, quarterly to tackle problems relating to radar.

- In India OR started in 1949, with the opening of a unit at the regional research laboratory, Hyderabad for using OR in planning and organizing research
- In 1953, an OR unit was formed at the Indian statistical institute, Calcutta with a view to apply OR techniques to national planning under the guidance of Prof. Mahalanobis.
- OR got its formal recognition in the country with the formation of OR society of India (ORSI) in 1957. The first president of the society was P.C. Mahalanobis.
- The society became the member of international federation of OR societies in 1959 and organized its first conference at all India level during the same year.
- The publication of journal "OPSEARCH" by ORSI in 1954 further added to the growth of OR activities in India.
- An international seminar "second OR around-the-world meeting" sponsored by ORSI was held at New-Delhi during August 7-10, 1967.

Today, however, service organizations such as banks, hospitals, libraries, airlines, railways etc. recognize the usefulness of OR in improving efficiency. Because of OR's multi-disciplinary character and application in varied fields, it has a bright future.

NATURE/CHARACTERISTICS OF OR

Operation Research can be considered as being the application of scientific method by interdisciplinary teams to problems involving the control of organized (men – machine) systems so as to provide solutions which are best for the purpose of the organization of a whole. Different characteristics constitute the nature of OR can be summed up as under:

Inter Disciplinary Team Approach

The OR has the characteristics that it is done by a team of scientists drawn from different disciplines such as mathematics, statistics, economics, engineering, physics etc., it is essentially an inter disciplinary

team approach. Each member of the OR team is benefited from the view point of the other so that a workable solution obtained through such collaborative study has a great chance of acceptance by management.

Systems Approach

OR emphasizes on the overall approach to the system. This characteristic of the OR is often referred as system oriented. The orientation is based on the observation that in the organized systems the behavior of any part ultimately has some effect on every other part. But all these effects are not significant and even not capable of detection. Therefore, the essence of system orientation lies in the systematic search for significant interactions in evaluating actions of any part of the organization.

OR can't give perfect answers or solutions to the problems it merely bad answers. To the problem which otherwise have worst answers. Thus, OR simply helps in improving the quality of the solution. But does not result into a perfect solution.

Goal Oriented Optimum Solution

OR tries to optimize a well-defined function subject to give constraints and as such as concerned with the optimization theory.

Scientific Method

OR involves scientific and systematic attack of complex problems to arrive at optimum solution. In other words, OR uses techniques of scientific research.

Thus OR comprehends both aspects i.e., it includes both scientific research on the phenomena at operating systems and the associated engineering activities aimed at applying the results of research.

Use of Models

OR use models built by quantitative measurement of the variable concerning a given problem and also derives a solution from the model using one or more of the diversified solution techniques. A solution may be extracted from a model either by conducting experiments on it or by mathematical analysis. The purpose help the management to determine its policy and actions scientifically.

DEFINITIONS OF OR

- OR is a scientific approach to problem solving for executive management
▪ **-H.M.WAGNER**
- OR may be described as a scientific approach to decision making that involves the operation of organizational system.
-F.S.HILLER AND G.J.LIBERMAN
- OR is a systematic application of quantitative methods, techniques and tools to the analysis of problems involving the operation of systems. **GEORGE**
- OR is essentially a collection of mathematical techniques and tools which in conjunction with a system approach, are applied to solve practical decision problems of an economic or engineering nature
-GEORGE

SCOPE/APPLICATIONS OF OR

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OR is today recognized as an applied science concerned with large number of diverse human activities. To be precise on operation uses some valuable recourses like man, money, machine, time, effort etc.

An OR worker is required,

- To minimize the input value for a specific output.
- To maximize the output value for a specific input
- To maximize some functions of these values

Example: The profit (or) return on investment

Some of the areas of management where techniques of OR are applied are listed below:

In Agriculture

With the explosion of population and consequent shortage of food, quantitative techniques are very useful in sorting out problems.

- The problem of optimum allocation of land in various crops in accordance with the climatic conditions.
- The problem of optimum distribution of water from various resources, fertilizers, new techniques and capabilities.

In Finance

In this modern times of economies crisis, it has become very necessary for every government to have careful planning using the quantitative techniques applied to:

- Maximize the per capita income with minimum resources.
- Find out the profit plan for the company.
- Determine the best replacement policies.

In Industry

If the industry manager decides his politics only on the basis of his past experiences, he may not lead the organization in a positive manner and organization will suffer heavy losses. This losses can be compensated by using quantitative techniques in business management with formulates total production schedule, optimum utilization of resources available

In Marketing

With the help of quantitative techniques, a marketing manager can decide where to distribute the product for sale so that the total carriage inwards and outwards cost will be minimize.

- Minimum per unit sale price.
- Size of the stock to meet the future demand.
- Selection of advertising media with to respect to time, cost, efforts, results etc.
- Purchasing criteria at minimum possible cost

In Personnel Management

A personnel manager can use quantitative techniques

- To appoint the most suitable persons an appropriate salary.
- To determine the best age of retirement.
- To find out the number of persons to be appointed on full time basis depending upon the nature of the work load.

In Production Management

A production manager can use quantitative techniques

- To find the number and size of the items to be produced.
- In scheduling and sequencing the production run by proper allocation of machines.
- In calculating the optimum product mix.
- To select, locate and design production plans
- ***In Insurance Sector***
 - What should be the premium rates for various modes of policies?
 - How best the profits could be distributed in the case of profit generation?

LIMITATIONS OF OR

The limitations of OR are mostly related with the problems concerned with model building and the time and cost factors involved in its application rather than its practical utility. Some of these limitations are:

- OR provides solution only when the problem consists of only quantifiable elements. But in real situation, all relevant variables or influencing factors cannot be qualified and some may be qualitative in nature. Factors which cannot be quantified cannot be used in the mathematical models used in OR.
- OR tries to find out the optimal solution, considering all the factors individual in a problem. In present day business problems are affected by a number of factors and expressing them with one another requires enormous calculations. Such calculations cannot be handled manually and require electronic computers involving heavy costs. This makes application of OR suitable to only very large organizations.
- Application of mathematical model of OR requires persons having knowledge of mathematics or statistics who might not be aware of the problems related to business. On the contrary, managers who can understand business problems fail to understand the complexity of OR. this creates a gap between the two
- Application of OR becomes a costly affair when the basic data is subjected to frequent changes. Also a fairly good solution which can be implement at present (i.e., satisfying solution) may be better than an optimal solution which may be possible to implement at a late data.
- Sometimes implementation of decisions becomes a delicate task because of resistance offered by people who are affected by the decision. Hence applications of OR models should take into account complexities of human relations and behavior.

PHASES OF OR

There are three phases of the scientific model on which OR approach is based.

Judgment Phase

This phase includes

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- Identification of the real life problem.
- Selection of an appropriate objective and selection of various variables related to this objective.
- Application of the appropriate scale of measurement i.e, measures of effectiveness.
- Formulation of an appropriate model of the problem.

Research Phase

This phase is the largest and longest among other two phases. This phase utilizes Observations and data collection for a better understanding of the problem.

- Formulation of hypothesis and model.
- Observation and experimentation to test the hypothesis.
- Analysis of the available information and verification of the hypothesis using pre-established measures of desirability.
- Prediction of various results from the hypothesis.
- Generation of the result and consideration of alternative method.

Action Phase

This phase consist of

- Making recommendations for implementing the decision by an individual who is in the position to implement results.
- This individual must be aware of the environment in which the problem occurred, objective, assumptions and omissions of the model of the problem.

LINEAR PROGRAMMING PROBLEM

INTRODUCTION

The term linear programming is a combination of two words, “*linear*” and “*programming*”. The term “*linear*” refers to linear relationships among the variables in a model. Thus, a given change in one variable is always cause a resulting proportional change in another variable. The word “*programming*” refers to modelling and solving problem mathematically that involves the economic allocation of limited resources by choosing a particular course of action or strategy among various alternative strategies to achieve the desire objectives.

Linear programming is a mathematical modelling technique useful for the allocation of “*source*” or “*limited*” resource, such as men (labor), machine, material, time, warehouses, capital etc., to several competing activities such as products, services, jobs, new projects on the basis of a given criterion of optimality.

Generally the criterion of optimality is return on investment, profit, cost, time, distance etc.,

In 1947, during world war-2, “George .B. Dantzing” while working with the US air force developed LP model, primarily for solving military logistics problems.

But now, it is being used extensional in all functional areas of management airlines, agriculture, military operations, education, transportation planning and research and development etc.,

STRUCTURE AND MODEL OF LPP

STRUCTURE

The genera structure of LPP consist of three basic components. They are

a: Decision Variables

The LPP can be applied to the given problem when the problem contains various alternatives [courses of actions] for arriving the optimal value of objective function. These alternatives are evaluated by nature of objective function and availability of resources. For this we study certain activities those are called “*decision variables*”. Usually denoted $x_1, x_2, \dots, x_n$.

Example:

In a product mix, contains 3 products viz., A, B and C and the number of units for product A is x_1 , B is x_2 and C is x_3 are the decision variables.

The values of these variables may or may not be under the decision makers, then such variables are said to be “*controllable*” otherwise “*uncontrollable*”

But in an LPP all decision variables are controllable continuous and non-negative i.e. $x_j \geq 0$; $j=1, 2, \dots, n$.

b: Objective Function

The objective function of each LPP is expressed in terms of decision variables to optimize [maximize or Minimize] the criterion of optimality such as profit, cost, revenue, distance etc.

It is represented as:

$$\begin{aligned}\text{Optimize [Maximize or Minimize]} Z &= \sum_{j=1}^n C_j X_j \\ &= c_1 x_1 + c_2 x_2 + \dots + c_n x_n\end{aligned}$$

Where Z= The optimality, which is a function of decision variables x_j ; $j=1,2,\dots$

C_j = Parameters that represents the contribution of a unit of the respective variables $x_1, x_2, \dots, x_n$

Note of Book:

- The optimal value of the given objective function is obtained by the graphical or simplex method.

c: Constraints

In every organization certain limitations or constraints are available on the use of resources viz., men, machine, material, capital etc. that limit the degree to which an objective can be achieved.

Such constraints must be expressed as linear equalities or inequalities in terms of decision variables.

Example:

Consider the labor constraint so as to use limited labor (50) for 3 products. For 3 products the decision variables are considered as x_1, x_2, x_3 and labor constraint is as: $x_1 + x_2 + x_3 \leq 50$

Note of Book:

- The solution of LPP must satisfy these constraints.

Mathematical Model of LPP

The general mathematical model of LPP contains 3 components as follows.

Objective Function:

$$\begin{aligned}\text{Optimize [Maximize or Minimize]} Z &= \sum_{j=1}^n C_j X_j \\ &= c_1 x_1 + c_2 x_2 + \dots + c_n x_n\end{aligned}$$

$$\begin{aligned}\text{STC:} \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \quad [\leq, =, \geq] b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \quad [\leq, =, \geq] b_2 \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \quad [\leq, =, \geq] b_i \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \quad [\leq, =, \geq] b_m\end{aligned}$$

Restrictions: $x_j \geq 0$; $j=1,2,\dots,n$

Note of Book:

- In any type of LPP, there will be 'n' number of decision variables and 'm' constraints.
- In LPP, the expression [$\leq$, $=$, $\geq$] means that in any specification problem each constraint may take only one of the three possible forms.
 1. $\leq$: Less than or equal to
 2. $\geq$: Greater than or equal to
 3. $=$: Is equals to

FORMULATION OF LPP

The following steps to be followed while formulating an LP Problem:

Step: 1: Identify the decision variables

Firstly, identify number of products 'n' and let the decision variables [$x_j \geq 0$; $j=1,2,\dots,n$] for n products.

Example:

A, B are two products in the given problem.

Let x_1 and x_2 are number of units of products A and B

Step: 2: identify the problem data

Identify the contribution [co-efficient] of decision variables for both objective function [C_j] and constraints [a_{ij}]

Step: 3: Formulate the objective function

Identify whether the objective function is to be maximized or minimized. Then formulate the objective function into a linear mathematical expression in terms of decision variables multiplied by their profit or cost contribution [co-efficient of decision variables, C_j]

$$\text{Maximize or Minimize } Z = \sum_{j=1}^n C_j X_j = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Step: 4: Formulate the constraints

Express the constraints verbally in terms of requirements, availability and equals to each resource.

The constraints are expressed by the resource availability as linear equality or inequality in terms of decision variables [x_j] multiplied by their profit or cost contribution [C_j] as follows:

- a. *At least as large as [minimum requirements]*
In this case the STC are expressed as greater than or equals to [$\geq$] and is used when the objective function is minimization.
- b. *No larger than [maximum availability]*

In this case the STC are expressed as less than or equals to [$\leq$] and is used when the objective function is maximization.

- c. *Exactly equals to*
- d. In this case the STC are expressed as equality [=] and is used when the objective function is both maximization and minimization.

Step: 5: Formulate LPP

Finally, formulate LPP as follows

Objective function: Maximize Z or Minimize $Z = \sum_{j=1}^n C_j X_j$

STC: $\sum_{j=1}^n a_{ij} x_j [\leq \text{ or } = \text{ or } \geq] b_i ; i=1,2,\dots,m$

Restrictions: $x_j \geq 0; j=1,2,\dots,n$

Assumptions of LPP

- The basic requirements of LPP is that both the objective and constraints must be expressed in terms of linear equalities and inequalities.
- The parameters and contribution must be known and fixed.
- The values of decision variables [x_j] must be non-negative i.e. $x_j \geq 0; j=1,2,\dots,n$

SOLUTION OF LPP

Linear programming problem can be solved by using any one of the following method.

1. Graphical Method
 - a. Maximization
 - b. Minimization
 - c. Mixed constraints
 - d. Unbounded solution
 - e. Infeasible solution
2. Simplex Method
 - a. Simple simplex method
 - b. Two-Phase method
 - c. Big-M method
 - d. Dual-Simplex method

Graphical Method

For LP Problems that have only two variables it is possible that the entire set of feasible solutions can be displayed graphically by plotting linear constraints on a graph paper to locate the best (optimal) solution. The technique used to identify the optimal solution is called the “*graphical solution approach or technique*” for an LP Problem with two variables i.e. x_1 and x_2 .

Important definitions

Solution: The set of values of decision variables $x_j (j=1,2,\dots,n)$ which satisfy the constraints of an LP Problem is said to constitute *solution* to that LP Problem.

Feasible Solution: The set of values of decision variables $x_j (j=1,2,\dots,n)$ which satisfy all the constraints and non-negativity conditions of an LP Problem simultaneously is said to constitute the *feasible solution* to that LP Problem.

Infeasible Solution: The set of values of decision variables $x_j (j=1,2,\dots,n)$ which do not satisfy the constraints and non-negativity conditions of an LP Problem simultaneously is said to constitute the *infeasible solution* to that LP Problem.

Basic Solution: For a set of 'm' simultaneous equations in 'n' variables ($n > m$), a solution obtained by setting (n-m) variables equal to zero and solving for remaining 'm' equations in 'm' variables is called a *basic solution*.

The (n-m) variables whose value did not appear in this solution are called non-basic variables and the remaining 'm' variables are called basic variables.

Basic Feasible Solution: A feasible solution to an LP Problem which is also the basic feasible solution is called the *basic feasible solution*. That is, all basic variables assume non-negative values. Basic feasible solutions are of two types:

- a. **Degenerate:** A basic feasible solution is called *degenerate* if value of at least one basic variable is zero.
- b. **Non-degenerate:** A basic feasible solution is called *non-degenerate* if values of all 'm' basic variables are non-zero and positive.

Optimal Basic Feasible Solution (OBFS): A basic feasible solution which optimizes (Maximization or Minimization) the objective function value of the given LP Problem is called *OBFS*.

Unbound Solution: A solution which can increase or decrease the value of objective function of the LP Problem indefinitely is called an *unbounded solution*.

GRAPHICAL ALGORITHM

The following steps are to be followed while using graphical method.

Step: 1: Formulation of LPP

Formulate the LP Problem in mathematically as follows:

$$\text{Maximize } Z \text{ or Minimize } Z = \sum_{j=1}^n C_j X_j$$

$$\text{STC: } \sum_{j=1}^n a_{ij} x_j [\leq \text{ or } = \text{ or } \geq] b_i ; i=1,2,\dots,m$$

$$\text{Restrictions: } x_j \geq 0; j=1,2,\dots,n$$

Step: 2: Construction of Graph

- Convert the inequality constraints into equalities which satisfies non-negative variables i.e. $x_1, x_2 \geq 0$.
- Determine the points of each given constraint. Randomly assign '0' value to x_1 and determine x_2 value and assign '0' value to x_2 and determine x_1 value for each given constraint.
- Now construct the graph by drawing horizontal and vertical lines which represented by X_1 -axis and X_2 -axis in the criterion X_1OX_2 , since all the points which satisfies the constraints must in the first quadrant only.
- Now plot these sets of values on the graph, connect these points by a straight line. Thus, there will be as many straight lines as there are equations, each straight line represents one constraint.

Step: 3: Identification of Feasible Solution

- For less than ($\leq$) constraints, the feasible area will be the area which lies below the straight lines towards the origin in the graph.
- For greater than ($\geq$) constraints, the feasible area will be the area which lies above the straight lines opposite to the origin in the graph.
- Finally, the feasible area will be the common area which satisfies all the constraints and restrictions.

Step: 4: Identification and determination of Extreme points

Identify the extreme or corner points of feasible area and determine the points using given equations.

Step: 5: Evaluation of Objective Function

Evaluate the objective function at various extreme points to get alternative solutions.

Step: 6: Determination of Optimal Value

The optimal value (maximization or Minimization) of objective function occurs at a particular point.

Simplex Method

The concept of simplex method is similar to the graphical method. In graphical method, extreme points of the feasible solution space are examined to search for optimal solution at one of them.

For LP Problem with several variable ($n > 0$) i.e. $x_1, x_2, \dots, x_n$, we may not be able to graph the feasible region. Thus, we need a more efficient method to suggest an optimal solution for such problems, called the "*simplex method*". The method was developed by G B Dantzing in 1947.

The simplex is an important term in mathematics that represents an object in n-dimensional space connecting (n+1) points. In one dimension, a simplex is a line segment connecting 2 points; in two dimensions, it is a triangle formed by joining 3 points; in three dimensions, it is a four sided pyramid having 4 points.

The simplex method examines the extreme points in a systematic manner, repeating the same set of steps of the algorithm until an optimal solution is reached. It is for this reason that it is also called the “*iterative method*”.

Mathematical form of an LP Problem

The standard form of the LP Problem is represented as follows:

Optimize [Maximize or Minimize] $Z = \sum_{j=1}^n C_j X_j$

$$= c_1 x_1 + c_2 x_2 + \dots + c_n x_n + 0s_1 + 0s_2 + \dots + 0s_n$$

$$\begin{aligned} \text{STC:} \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + s_2 = b_2 \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n + s_i = b_i \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + s_m = b_m \end{aligned}$$

Restrictions: $x_j \geq 0; j=1, 2, \dots, n$

Note of Book:

- The objective function should be of either the *maximization* or *minimization* type.
- The right hand side of each constraint should be made non-negative; if it is not, this should be done by multiplying both sides of the resulting constraint by ‘-1’.
- All the constraints should be expressed as equations by adding additional variables.

Additional Variables

Additional variables are added in the given LP Problem to convert it into the standard form for the following reasons:

- These variables allow us to convert inequalities into equalities, thereby converting the given LP Problem into a form that is amenable to algebraic solution.
- These variables permit us to make a more comprehensive economic interpretation of a final solution.
- Help us to get an initial feasible solution represented by the columns of the identity matrix.

The three types of additional variables, viz.,

- a. *Slack variables(s)*: A slack variable represents unused resource, either in the form of time on a machine, labor hours, money, warehouse space or any number of such resources in various business problems. Since these variables yield on profit, these such variables are added to the original object function with zero coefficients.

- b. *Surplus variables(s)*: A surplus variable represents amount by which solution values exceed a resource. These variables are also called negative slack variables. Surplus variables, like slack variables carry a zero coefficient in the objective function.
- c. *Artificial variables(A)*: Artificial variables have no meaning in a physical sense and are only used as a tool for generating an initial LP Problem solution.
Artificial variables are added to the original objective function with ‘-M’ coefficients [Max Z] and ‘+M’ coefficients [Min Z]

Simple Simplex Method [maximization]

Simple simplex method is used to solve the LP Problem with maximization objective. The steps of the simplex method to obtain an optimal solution (if it exists) to an LP Problem are as follows:

SIMPLEX ALGORITHM

Step: 1: Formulation of LPP

To formulate the LPP follow the below given rules:

- Formulate the mathematical model of the given LPP.
- Check whether all the b_i [$i=1,2,\dots,m$] values are positive i.e. $b_i > 0$.
- Add additional variables to the given each constraint to convert into equalities.

(or)

Convert the inequality constraints into equalities by adding additional variables to the left hand side of each constraint and assign coefficients to those in objective function.

Note of Book:

- The summary of the additional variables to be added in the given LP Problem to convert it into a standard form is given in the following table.

Sl. No	Type of constraint	Type of variable	Type of objective function	
			Maximize Z	Minimize Z
1	Less than or equal to ($\leq$)	A variable is added	0	0
2	Greater than or equal to ($\geq$)	A surplus variable is subtracted	0	0
		An artificial variable is added	-M	+M
3	Equals to ($=$)	An artificial variable is added	-M	+M

- Before getting the final simplex solution, all artificial variables must be dropped out from the table.
- Consider all additional variables as basic variables.

Step: 2: Setup the initial solution

Write down the coefficients of all the variables in the LP model in the tabular form as shown below:

Initial simplex table

C_B	B	C_j	C_1	C_2	C_3		C_n	0	0	0.....0
		$b_i = X_B$	X_1	X_2	X_3		X_n	S_1	S_2	S_3 S_m
$(0)C_{B1}$	S_1	b_1	a_{11}	a_{12}	a_{13}		a_{1n}	1	0	0.....0
$(0)C_{B2}$	S_2	b_2	a_{21}	a_{22}	a_{23}		a_{2n}	0	1	0.....0
.	.	.	.	.	.		.	.	.	
.	.	.	.	.	.		.	.	.	
.	.	.	.	.	.		.	.	.	
$(0)C_{Bm}$	S_m	b_m	a_{m1}	a_{m2}	a_{m3}		0	0	0	1
$Z_j = \sum C_{Bi} X_j$			0	0	0		0	0	0	0.....0
$C_j - Z_j$			C_1	C_2	C_3		C_n	0	0	0.....0

Where,

C_B = Coefficient of Basic variables

B = Basic variable

b_1 = value of basic variable when $X_j = 0$

C_j = Incoming unit (profit or cost) or coefficient of decision variables in objective function

Z_j = Outgoing unit (profit or cost)

$C_j - Z_j$ = Net profit or cost

Step: 3: Test for optimality

Calculate the $(C_j - Z_j)$ value for all non-basic variables. Examine the values of $(C_j - Z_j)$ as follows:

- If all $(C_j - Z_j) \leq 0$, then the basic feasible solution is optimal.
- If at least one $(C_j - Z_j) > 0$, then it indicates that an improvement in the value of objective function Z is possible.

Step: 4: Test for feasibility (variable to leave the basis)

Select an entering variable to leave the basic variable by using the following steps:

- Select a column which has largest positive $(C_j - Z_j)$ value and called as key column.
- Calculate minimum ratio for each row.

$$MR = \frac{b}{kce} = \frac{b}{\text{key column element}}$$

- c. Select a row which has minimum positive value of MR and called as key row.
- d. Select an intersect element of key column and key row and called as key element.
- e. Select the leaving element [basic variable] in key row and replace it by an entering element [decision variable] from key column.

Step: 5: Finding the new solution

The new solution can be calculated based on rows and the new values can be calculated for 'b' and X_j , and s_i columns as follows:

Key row changes:

- a. If the key element is '1', then the key row remains the same in the new simplex table.
- b. If the key element is other than 1, then divide each key row element by the key element, i.e.

$$\text{Key row changes} = \frac{\text{key row element}}{\text{key element}}$$

Other row changes (ORC):

- a. $\text{ORC} = (\text{old row element in that row}) - (\text{corresponding Key row element} * \text{Fixed Ratio})$
 $\text{Fixed Ratio} = \frac{\text{key column element}}{\text{key element}}$
- b. Formulate new simplex table by using new values

Step: 6: Repeat the procedure

Go to step: 3 and repeat the procedure until all $(C_j - Z_j)$ values are either negative or zero i.e. $C_j - Z_j \leq 0$

Linear programming problem [LPP]

Optimisation: profit increase, cost increase-decrease

Mathematical model of LPP:

The General mathematical model of LPP contains three components as follows:

1) Objective function: optimize [maximize (or) minimize]

$$Z = \sum_{j=1}^n C_j x_j = C_1 x_1 + C_2 x_2 + \dots + C_n x_n$$

2) Subject to constraints:

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n [\leq, =, \geq] b_1$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n [\leq, =, \geq] b_2$$

$$\vdots$$
$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n [\leq, =, \geq] b_i$$

$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n [\leq, =, \geq] b_m$$

3) NON-Negativity Restrictions:

$$x_j \geq 0; \quad j=1, 2, 3, \dots, n$$

1) Objective function: $\text{Max } Z = 20x_1 + 50x_2$

$$2) \text{STC} : 2x_1 + 5x_2 \leq 50$$

$$4x_1 + 3x_2 \leq 10$$

3) NNR : $x_1, x_2 \geq 0$

max = profit
min = cost

LPP formulation:

1) A company manufacture two types of products 'P₁' and 'P₂' each product use ^(machine name) lathe & milling machines the processing time per unit of 'P₁' on the lathe 5hrs on the milling machine is 4hrs. The processing time per unit of 'P₂' on lathe is 10hrs & on the milling machine is 4hrs. The maximum No of hours available per week on the lathe & milling machines are 60hrs and 40hrs respectively. Also the profit per unit of selling P₁ & P₂ are 6Rs & 8Rs. Formulate a linear programming model to determine the production volume of each of the product such that the total profit is maximized.

Sol

① $\text{Max } Z = 6x_1 + 8x_2$

② STC:

lathe: $- 5x_1 + 10x_2 \leq 60$

milling: $- 4x_1 + 4x_2 \leq 40$

③ NNR:

$x_1, x_2 \geq 0$

Machine type	processing type		Available hrs
	P ₁	P ₂	
Lathe	5	10	60
milling	4	4	40
Profit	₹6	₹8	

2) 'A' & B are two products to be manufactured unit profit on 40Rs. & 35Rs. Maximum material available is 60kg, and labour 96hrs. If each unit of 'A' needs 2kg of material & 3 man hrs, whereas each unit of 'B' needs 4kg of material & 3 man hrs. Find optimal level of 'A' & 'B' to be manufactured formulate the LPP.

Sol

Objective function:

$$\text{Max } Z = 40x_1 + 35x_2$$

STC $M = 60x_1 + 2x_2 \leq 4 \text{ kg}$

$$L = 96x_1 + 3x_2 \leq 3 \text{ kg}$$

NLR

$$x_1, x_2 \geq 0$$

Products	P.T		hrs.
	A	B	
material	60	2	4
labour	96	3	3

3) A manufacture produces two types of models m_1 & m_2 . Each model of m_1 requires 4hrs of polishing & 2hrs of grinding. Where each of the model m_2 requires 2hrs of polishing & 5hrs of grinding. The manufacture have 2 polishes & 3 grinders. Each polisher works 40hrs a week & each grinder works 60 hrs a week. Profit & m_1 is 3Rs. & m_2 is 4Rs. Formulate the LPP.

Sol Objective function:

$$3x_1 + 4x_2$$

STC:

polishing:

$$4x_1 + 2x_2 \leq 2 \times 40 = 80$$

$$2x_1 + 5x_2 \leq 3 \times 60 = 180$$

NLR:

$$x_1, x_2 \geq 0$$

products	Preceding type		hrs
	m_1	m_2	
P	4	2	80
g	2	5	180

Solution of linear programming:

LPP can be solved basically by three ways.

→ Graphical Method

→ Simplex Method

→ Transportation Method

Graphical Method:

Solving LPP using Graphical Method

Procedure:

- The industrial problems involving 2 or 3 variables can be easily & effectively solved.
- By drawing the graph for various constraints & Objective function.
- It is even simple for 2 variable cases. As three dimensional geometric is either rather difficult to draw on graph paper.

Steps for Graphical solution of LPP:

- * The constraints of the problem are taken as equalities.

and plotted on the graph paper. The region satisfy for the each constrain is determine & then the region to common they all constrains. Relations is located. This region is known as feasible region for the problem.

* As all points lying in this region will simultaneously satisfy the given constrains.

→ The solution set of the system of constrains is found to be represent by sum polygon term turned as a feasible region.

Step 2:

The objective function is also plotted on the same Graph paper by taking same arbitrary value to 'Z'.

Step 3:

Try & Error method is used to locate the optimal solution inside the feasible region determined in step 1. So that the objective function is maximized it is experienced that the optimal solution is located near the extreme point of that feasible region. Thus the optimal feasible point should be such only at the corners of the converse convex polygon. The graph for all the constrains & objective function will be straight lines in LPP.

19/6/2023

Problems on Graphical Method:

Maximization problem with 2 constraints:

1) solve the LPP by using Graphical method:

$$\text{Max } Z = 12x_1 + 16x_2 \text{ [objective function]}$$

Subject to constraints:

$$10x_1 + 20x_2 \leq 120$$

$$8x_1 + 8x_2 \leq 80$$

Non-Negativity Restrictions:

$$x_1, x_2 \geq 0$$

Sol Given equations are:

$$10x_1 + 20x_2 = 120 \rightarrow (1)$$

$$8x_1 + 8x_2 = 80 \rightarrow (2)$$

from eq (1)

$$10x_1 + 20x_2 = 120$$

when $x_1 = 0$ (assume)

$$10(0) + 20x_2 = 120$$

$$0 + 20x_2 = 120$$

$$x_2 = \frac{120}{20} = 6$$

$$\boxed{x_2 = 6}$$

when $x_2 = 0$ [it can assume]

$$10x_1 + 20x_2 = 120$$

$$10x_1 + (0) = 120$$

$$x_1 = \frac{120}{10} = 12$$

$$\boxed{x_1 = 12}$$

x_1	0	12
x_2	6	0

(0,6) (12,0)

From eq ②

$$8x_1 + 8x_2 = 80 \rightarrow ②$$

when $x_1 = 0$ [it can assume the value]

$$8(0) + 8x_2 = 80$$

$$x_2 = \frac{80}{8}$$

$$\boxed{x_2 = 10}$$

x_1	0	10
x_2	10	0

when $x_2 = 0$

$$8x_1 + 8x_2 = 80$$

$$8x_1 + (0) = 80$$

$$x_1 = \frac{80}{8}$$

$$\boxed{x_1 = 10}$$

From the graph

ABCD is a feasible region

A(0,0), B(10,0), C(8,2), D(0,6)

$$\text{Max } Z = 12x_1 + 16x_2$$

$$\text{Now } A(0,0)$$

$$= 12(0) + 16(0)$$

$$= 0 + 0$$

$$= 0$$

$$\max Z = 12x_1 + 16x_2$$

$$B = (10, 0)$$

$$12(10) + 16(0)$$

$$= 120 + 0$$

$$= 120$$

$$\max Z = 12x_1 + 16x_2$$

$$C = (8, 2)$$

$$= 12(8) + 16(2)$$

$$= 96 + 32$$

$$= 128$$

$$\max Z = 12x_1 + 16x_2$$

$$D = (0, 6)$$

$$= 12(0) + 16(6)$$

$$= 0 + 96$$

$$= 96$$

$$\therefore Z(\text{optimization}) = 128 \text{ at } (8, 2)$$

(max)

$$x_1 = 8, x_2 = 2$$

2) maximization problem with 2 constraints

① solve the LPP by using Graphical method!

$$\max Z = 40x_1 + 35x_2$$

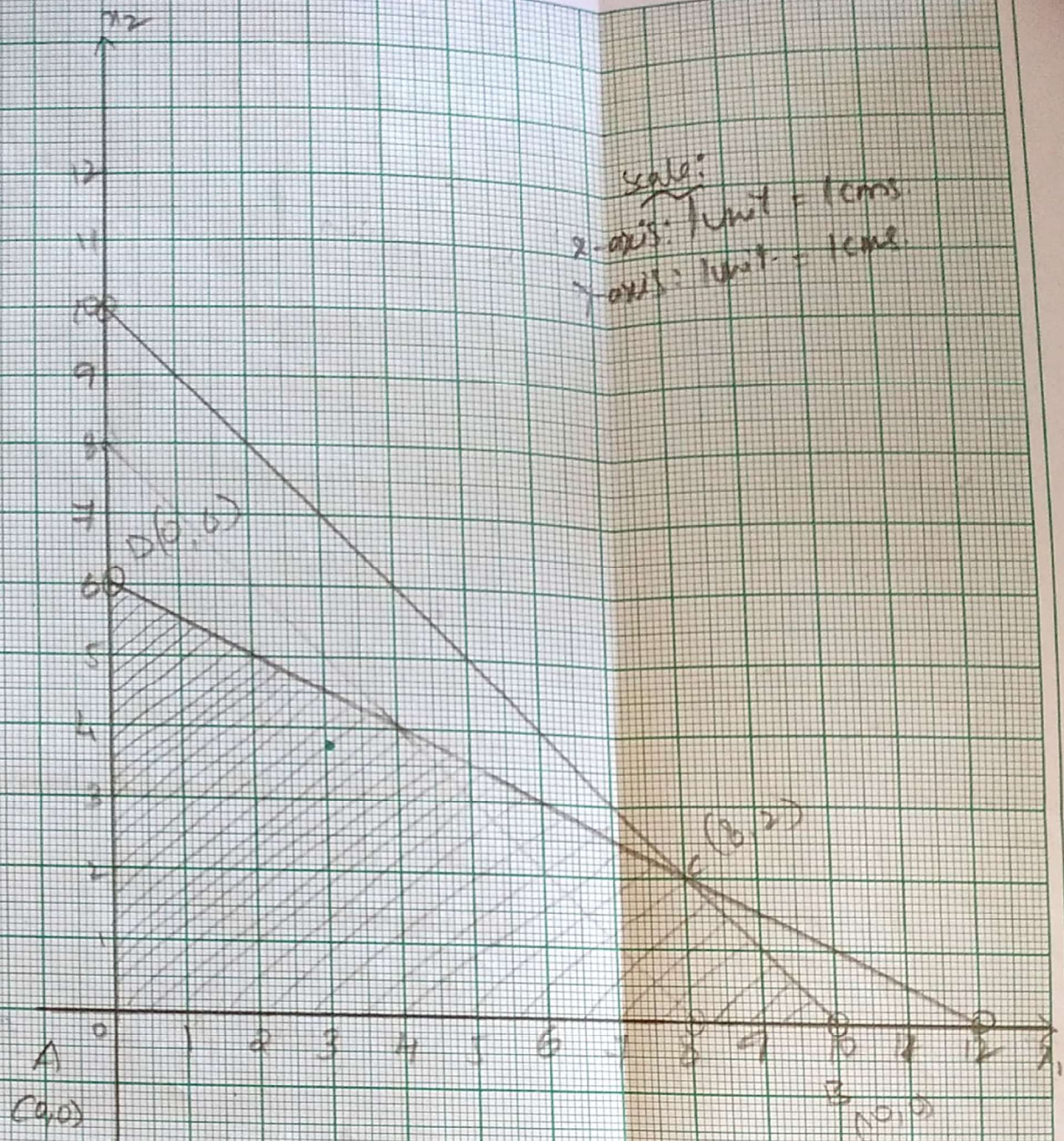
② S.T.C:-

$$2x_1 + 4x_2 \leq 60$$

$$3x_1 + 3x_2 \leq 96$$

③ MINR:

$$x_1, x_2 \geq 0$$



2) maximization prblm with 2 constraints

① solve the LPP by using Graphical method!

$$\max Z = 40x_1 + 35x_2$$

② S.T.C:-

$$2x_1 + 4x_2 \leq 60$$

$$3x_1 + 3x_2 \leq 96$$

③ NR:

$$x_1, x_2 \geq 0$$

Sol Given

$$2x_1 + 4x_2 = 60 \rightarrow (1)$$

$$3x_1 + 3x_2 = 96 \rightarrow (2)$$

from eq (1)

$$2x_1 + 4x_2 = 60$$

$$\text{where } x_1 = 0$$

$$2(0) + 4x_2 = 60$$

$$0 + 4x_2 = 60$$

$$x_2 = \frac{60}{4} = 15$$

$$\boxed{x_2 = 15}$$

where $x_2 = 0$

$$2x_1 + 4(0) = 60$$

$$2x_1 + 0 = 60$$

$$x_1 = \frac{60}{2} = 30$$

$$\boxed{x_1 = 30}$$

from eq (2)

$$3x_1 + 3x_2 = 96$$

where $x_1 = 0$

$$3(0) + 3x_2 = 96$$

$$0 + 3x_2 = 96$$

$$x_2 = \frac{96}{3} = 32$$

$$\boxed{x_2 = 32}$$

where $x_2 = 0$

$$3x_1 + 3(0) = 96$$

$$3x_1 + 0 = 96$$

$$x_1 = \frac{96}{3} = 32$$

$$\boxed{x_1 = 32}$$

x_1	0	30
x_2	15	0

(0, 15) (30, 0)

x_1	0	32
x_2	32	0

(0, 32) (32, 0)

from the graph

ABC is a feasible region

$$A(0,0) \quad B(30,0) \quad C(0,15)$$

$$\text{Max } z = 40x_1 + 35x_2$$

$$= 40(0) + 35(0)$$

$$= 0 + 0$$

$$= 0$$

$$\text{Max } z = 40x_1 + 35x_2$$

$$= 40(30) + 35(0)$$

$$= 1200 + 0$$

$$= 1200$$

$$\text{Max } z = 40x_1 + 35x_2$$

$$= 40(0) + 35(15)$$

$$= 0 + 525$$

$$= 525$$

∴ optimization of max $z = 1200$ is at in ~~(0,0)~~ (30,0)

8	0	1
0	3	1

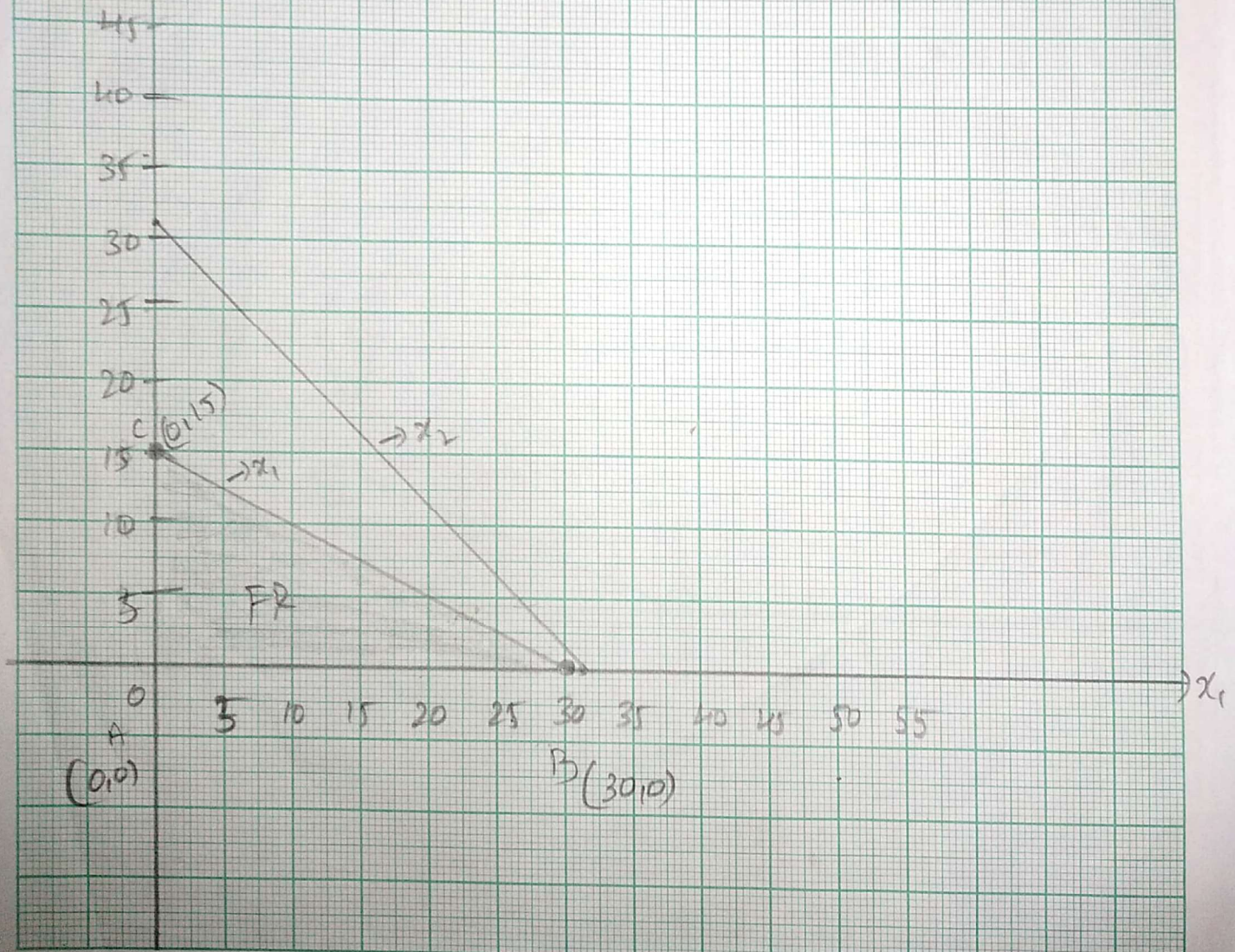
② maximum
 x_2

Vidya

Scale:

$$x_1 = 1 \text{ cm} = 5 \text{ units}$$

$$x_2 = 1 \text{ cm} = 5 \text{ units}$$



3) max Z problem with 3 constraints

Solve the LPP by using graphical method

Objective function:

$$\max Z = 5x_1 + 7x_2$$

STC

$$3x_1 + 8x_2 \leq 24$$

$$x_1 + x_2 \leq 4$$

$$10x_1 + 7x_2 \leq 35$$

NNC

$$x_1, x_2 \geq 0$$

Sol Given

$$3x_1 + 8x_2 = 24 \rightarrow (1)$$

We can assume $x_1 = 0$

$$3(0) + 8x_2 = 24$$

$$8x_2 = 24$$

$$x_2 = \frac{24}{8}$$

$$x_2 = 3$$

x_1	0	8
x_2	3	0

(0,3) (8,0)

Now $x_2 = 0$

$$3x_1 + 8x_2 = 24$$

$$3x_1 + 8(0) = 24$$

$$3x_1 = 24$$

$$x_1 = \frac{24}{3}$$

$$x_1 = 8$$

eq (2)

$$x_1 + x_2 = 4$$

Now $x_1 = 0$ it can assume

$$0 + x_2 = 4$$

$$\boxed{x_2 = 4}$$

Now $x_2 = 0$

$$x_1 + 0 = 4$$

$$\boxed{x_1 = 4}$$

x_1	0	4
x_2	4	0

(0,4) (4,0)

eq (3)

$$10x_1 + 7x_2 = 35$$

Now $x_1 = 0$

$$10(0) + 7x_2 = 35$$

$$0 + 7x_2 = 35$$

$$x_2 = \frac{35}{7}$$

$$\boxed{x_2 = 5}$$

" $x_2 = 0$ "

$$10x_1 + 7x_2 = 35$$

$$10x_1 + 7(0) = 35$$

$$10x_1 + 0 = 35$$

$$x_1 = \frac{35}{10}$$

$$\boxed{x_1 = 3.5}$$

From the graph

A B C D E is a feasible region

A(0,0)

$$A(0,0) \quad B(3.5,0) \quad C(2.5,1.7) \quad D(1.6,2.4) \quad E(0,3)$$

$$\max z = 5x_1 + 7x_2$$

$$A(0,0)$$

$$5(0) + 7(0)$$

$$= 0 + 0$$

$$= 0$$

$$\max z = 5x_1 + 7x_2$$

$$= 5(3.5) + 7(0)$$

$$= 17.5$$

$$\max z = 5x_1 + 7x_2$$

$$5(2.5) + 7(1.7)$$

$$= 12.5 + 11.9$$

$$= 24.4$$

$$\max z = 5x_1 + 7x_2$$

$$= 5(1.6) + 7(2.4)$$

$$= 8 + 16.8$$

$$= 24.8$$

$$\max z = 5x_1 + 7x_2$$

$$5(0) + 7(3)$$

$$= 0 + 21$$

$$= 21$$

optimization of $\max z = 24.4$ as it on
 $(2.5)(1.7)$

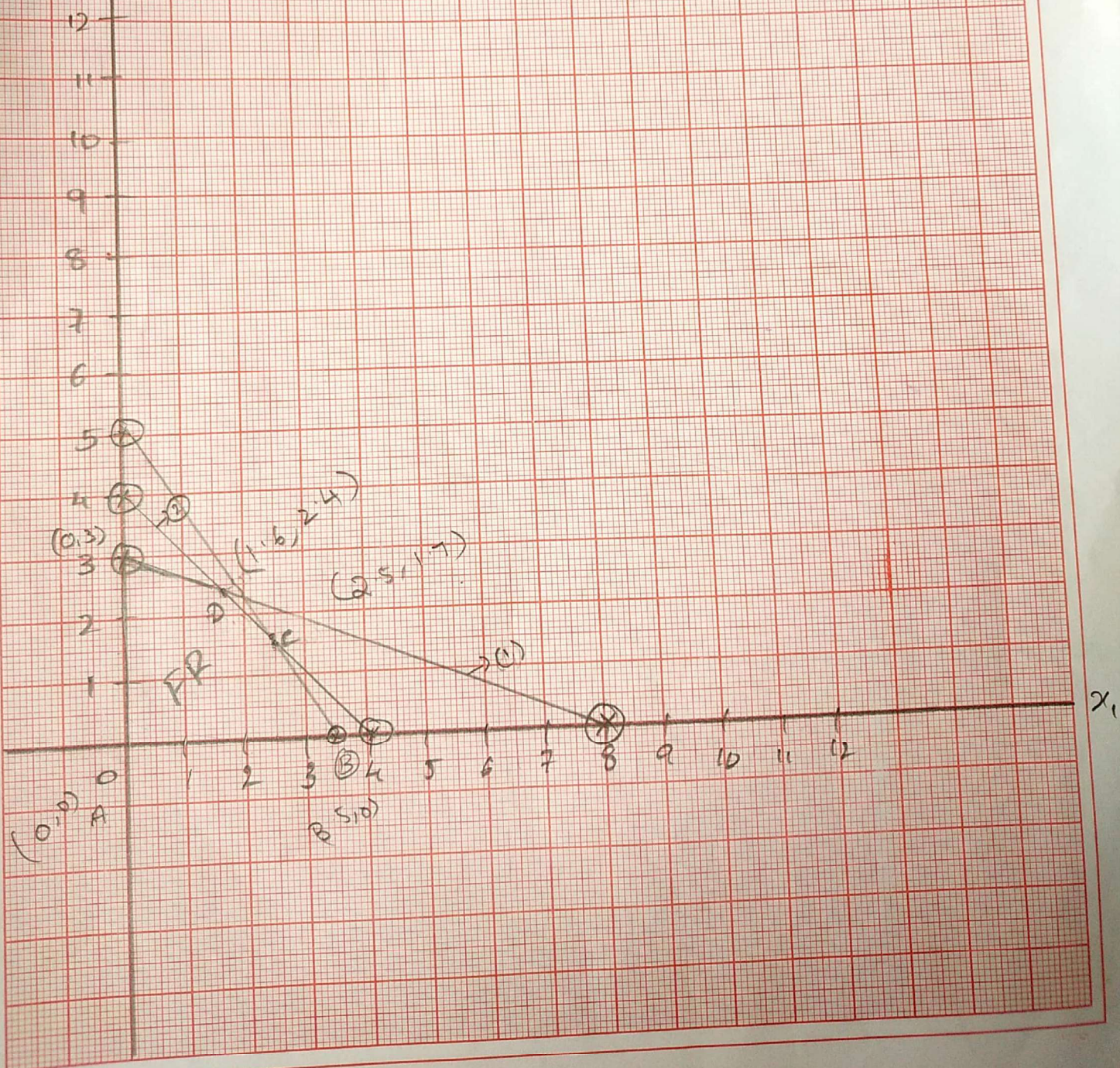
maximum
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Vidhya
Page No.: _____

Scale:

$$x_1 = 1\text{cm} = 1\text{unit}$$

$$x_2 = 1\text{cm} = 1\text{unit}$$



1) minimization prblm with 2 constrains
 solve the following LPP by using graphical method.

$$\min Z = 4x_1 + 6x_2 \text{ [objective function]}$$

STC:

$$x_1 + x_2 \geq 8$$

$$6x_1 + x_2 \geq 12$$

NNR

$$x_1, x_2 \geq 0$$

Sol Given

$$x_1 + x_2 \geq 8$$

$$\text{Now } x_1 = 0$$

$$0 + x_2 = 8$$

$$\boxed{x_2 = 8}$$

$$\text{Now } x_2 = 0$$

$$x_1 + 0 = 8$$

$$\boxed{x_1 = 8}$$

x_1	0	8
x_2	8	0
	(0,8)	(8,0)

from eq $\rightarrow$ ②

$$6x_1 + x_2 = 12$$

$$\text{Now } x_1 = 0$$

$$6(0) + x_2 = 12$$

$$0 + x_2 = 12$$

$$\boxed{x_2 = 12}$$

$$\text{Now } x_2 = 0$$

$$6x_1 + 0 = 12$$

$$6x_1 = 12$$

$$x_1 = \frac{12}{6}$$

$$\boxed{x_1 = 2}$$

x_1	0	2
x_2	12	0
	(0,12)	(2,0)

From the graph

$$4x_1 + 6x_2$$

A, B, C is a feasible region

$$A(8,0), B(0.8, 7.2), C(0,12)$$

$$\min$$

$$\max z = 4x_1 + 6x_2$$

$$A(8,0)$$

$$4(8) + 6(0)$$

$$= 32 + 0$$

$$= 32$$

$$\min$$

$$\max z = 4x_1 + 6x_2$$

$$= 4(0.8) + 6(7.2)$$

$$= 3.2 + 43.2$$

$$= 46.4$$

$$\min z = 4x_1 + 6x_2$$

$$= 4(0) + 6(12)$$

$$= 0 + 72$$

$$= 72$$

⇒ optimization $\min z = 32$ as it is in $(8,0)$ [least value]

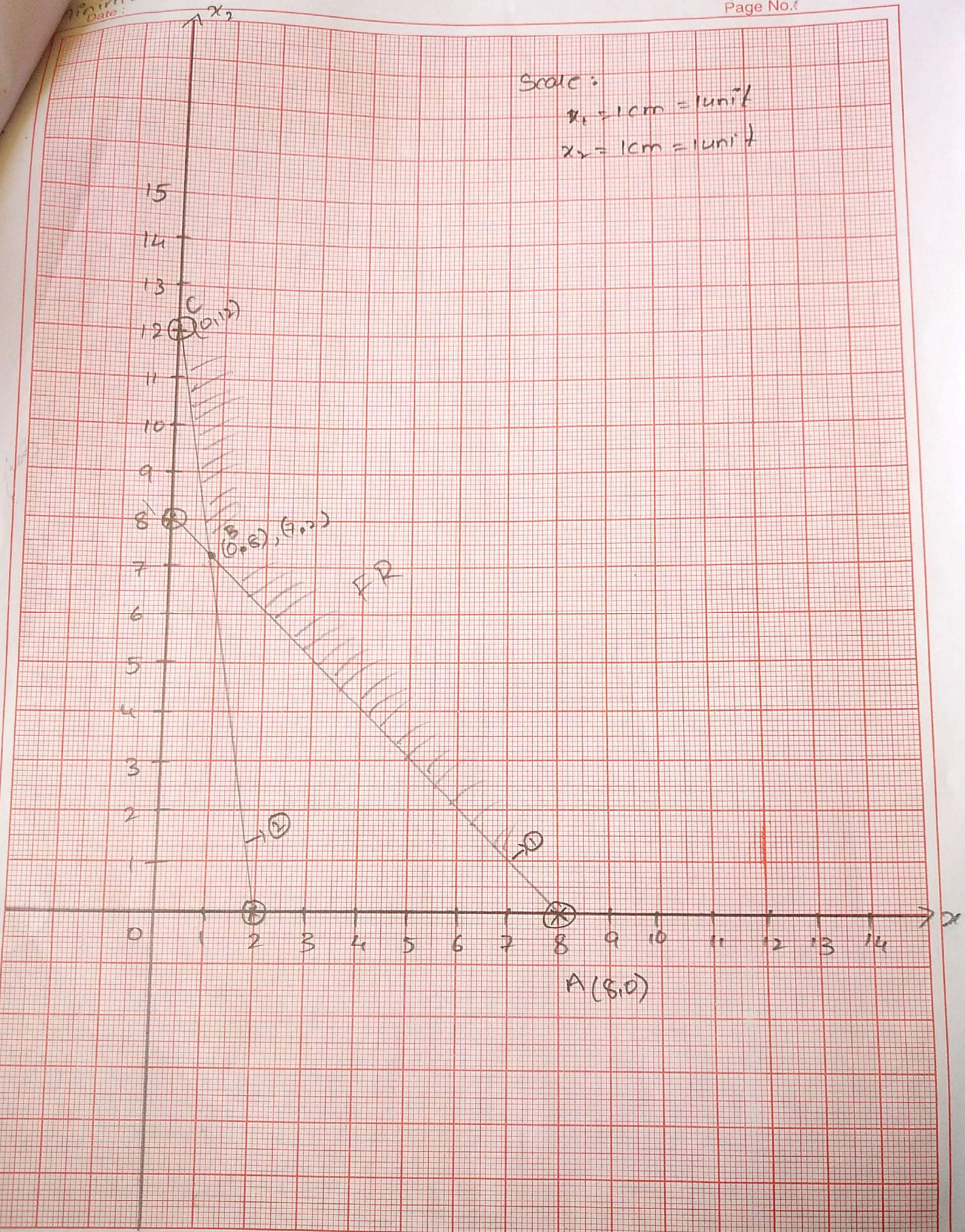
8	0	x_1
0	8	x_2
(8,0)	(0,8)	

8	0	x_1
0	8	x_2
(8,0)	(0,8)	

Scale:

$x_1 = 1 \text{ cm} = 1 \text{ unit}$

$x_2 = 1 \text{ cm} = 1 \text{ unit}$



2) minimization problem with 3 constraints

$$\min Z = 3x_1 + 2x_2 \quad [\text{Objective function}]$$

STC

$$5x_1 + x_2 \geq 10$$

$$x_1 + x_2 \geq 6$$

$$x_1 + 4x_2 \geq 12$$

NNC

$$x_1, x_2 \geq 0$$

Sol Given

$$5x_1 + x_2 = 10$$

$$\text{Where } x_1 = 0$$

$$5(0) + x_2 = 10$$

$$0 + x_2 = 10$$

$$\boxed{x_2 = 10}$$

$$\text{Where } x_2 = 0$$

$$5x_1 + 0 = 10$$

$$x_1 = \frac{10}{5} = 2$$

$$\boxed{x_1 = 2}$$

from eq → ②

$$x_1 + x_2 = 6$$

$$x_1 = 0$$

$$0 + x_2 = 6$$

$$\boxed{x_2 = 6}$$

$$\text{Now } x_2 = 0$$

$$x_1 + 0 = 6$$

$$\boxed{x_1 = 6}$$

x_1	0	2
x_2	10	0

(0,10) (2,0)

x_1	0	6
x_2	6	0

(0,6) (6,0)

from eqⁿ (3)

$$x_1 + 4x_2 = 12$$

where $x_1 = 0$

$$0 + 4x_2 = 12$$

$$x_2 = \frac{12}{4}$$

$$x_2 = 3$$

x_1	0	12
x_2	3	0

(0,3) (12,0)

where $x_2 = 0$

$$x_1 + 4(0) = 12$$

$$x_1 + 0 = 12$$

$$x_1 = 12$$

x_1	0	12
x_2	0	3

(0,0) (12,0)

from the above graph

ABCD is a feasible region

A(0,0) B(1,5) C(4,2) D(0,12)

$$\text{Min } z = 3x_1 + 2x_2$$

$$= 3(0) + 2(0)$$

$$= 0 + 0$$

$$= 0$$

$$\text{min } z = 3x_1 + 2x_2$$

$$= 3(1) + 2(5)$$

$$= 3 + 10$$

$$= 13$$

$$\text{min } z = 3x_1 + 2x_2$$

$$= 3(4) + 2(2)$$

$$= 12 + 4$$

$$= 16$$

$$\text{min } z = 3x_1 + 2x_2$$

$$= 3(0) + 2(12) \quad 3(12) + 2(0)$$

$$= 0 + 24$$

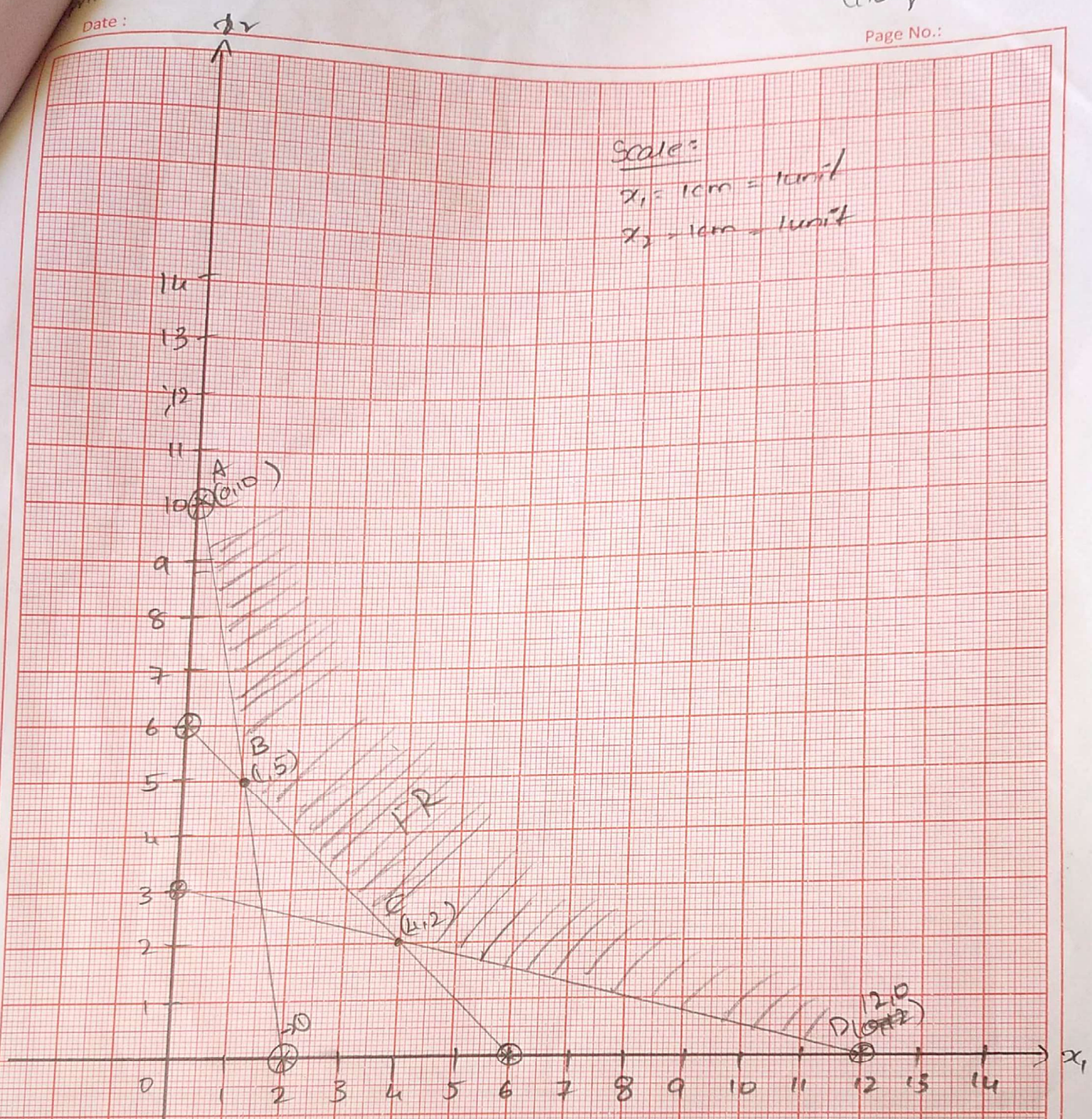
$$= 24$$

$$= 36$$

Optimization at (1,5) is in

13 value.

Scale:
 $x_1 = 1\text{cm} = 1\text{unit}$
 $x_2 = 1\text{cm} = 1\text{unit}$



3) Minimization problem with 3 constraints
Solve the LPP by using graphical method

$$\min Z = 20x_1 + 10x_2$$

STC

$$x_1 + 2x_2 \leq 40$$

$$3x_1 + x_2 \geq 30$$

$$4x_1 + 3x_2 \geq 60$$

NMC

$$x_1, x_2 \geq 0$$

Sol Given

$$x_1 + 2x_2 = 40$$

$$\text{Now } x_1 = 0$$

$$0 + 2x_2 = 40$$

$$x_2 = \frac{40}{2}$$

$$\boxed{x_2 = 20}$$

$$\text{Now } x_2 = 0$$

$$\bullet x_1 + 2(0) = 40$$

$$x_1 + 0 = 40$$

$$\boxed{x_1 = 40}$$

from eq ②

$$3x_1 + x_2 = 30$$

$$x_1 = 0$$

$$3(0) + x_2 =$$

$$3(0) + x_2 = 30$$

$$0 + x_2 = 30$$

$$\boxed{x_2 = 30}$$

x_1	0	40
x_2	20	0

(0,20) (40,0)

x_1	0	10
x_2	30	0

(0,30) (10,0)

$$3x_1 + x_2 = 30$$

$$3x_1 + (0) = 30$$

$$x_1 = \frac{30}{3}$$

from eq $\rightarrow$ (3)

$$4x_1 + 3x_2 = 60$$

$$x_1 = 0$$

$$4(0) + 3x_2 = 60$$

$$0 + x_2 = \frac{60}{3}$$

$$\boxed{x_2 = 20}$$

$$x_2 = 0$$

$$4x_1 + 3(0) = 60$$

$$4x_1 + 0 = 60$$

$$x_1 = \frac{60}{4}$$

$$\boxed{x_1 = 15}$$

x_1	0	15
x_2	20	0

From the above graph

A B C D is a feasible region

A(40,0) B(4,18) C(0,30) D(6,12)

$$\min Z = 20x_1 + 10x_2$$

$$20(40) + 10(0)$$

$$= 800 + 0$$

$$= 800$$

$$\begin{aligned}\min z &= 20x_1 + 10x_2 \\ &= 20(4) + 10(18) \\ &= 80 + 180 \\ &= 260\end{aligned}$$

$$\begin{aligned}\min z &= 20x_1 + 10x_2 \\ &= 20(0) + 10(30) \\ &= 0 + 300 \\ &= 300\end{aligned}$$

$$\begin{aligned}\min z &= 20x_1 + 10x_2 \\ &= 20(6) + 10(12) \\ &= 120 + 120 \\ &= 240\end{aligned}$$

Optimization of min z is ~~300~~ 240 at (6, 12)

z	0	8	18
x_1	0	30	18

$$\begin{aligned}d &= 20x_1 + 10x_2 \\ d &= 20(4) + 10(18) \\ d &= 80 + 180 \\ d &= 260\end{aligned}$$

$$d = 18$$

$$d = 20x_1 + 10x_2$$

$$d = 20(0) + 10(30)$$

$$d = 0 + 300$$

$$d = 20(6) + 10(12)$$

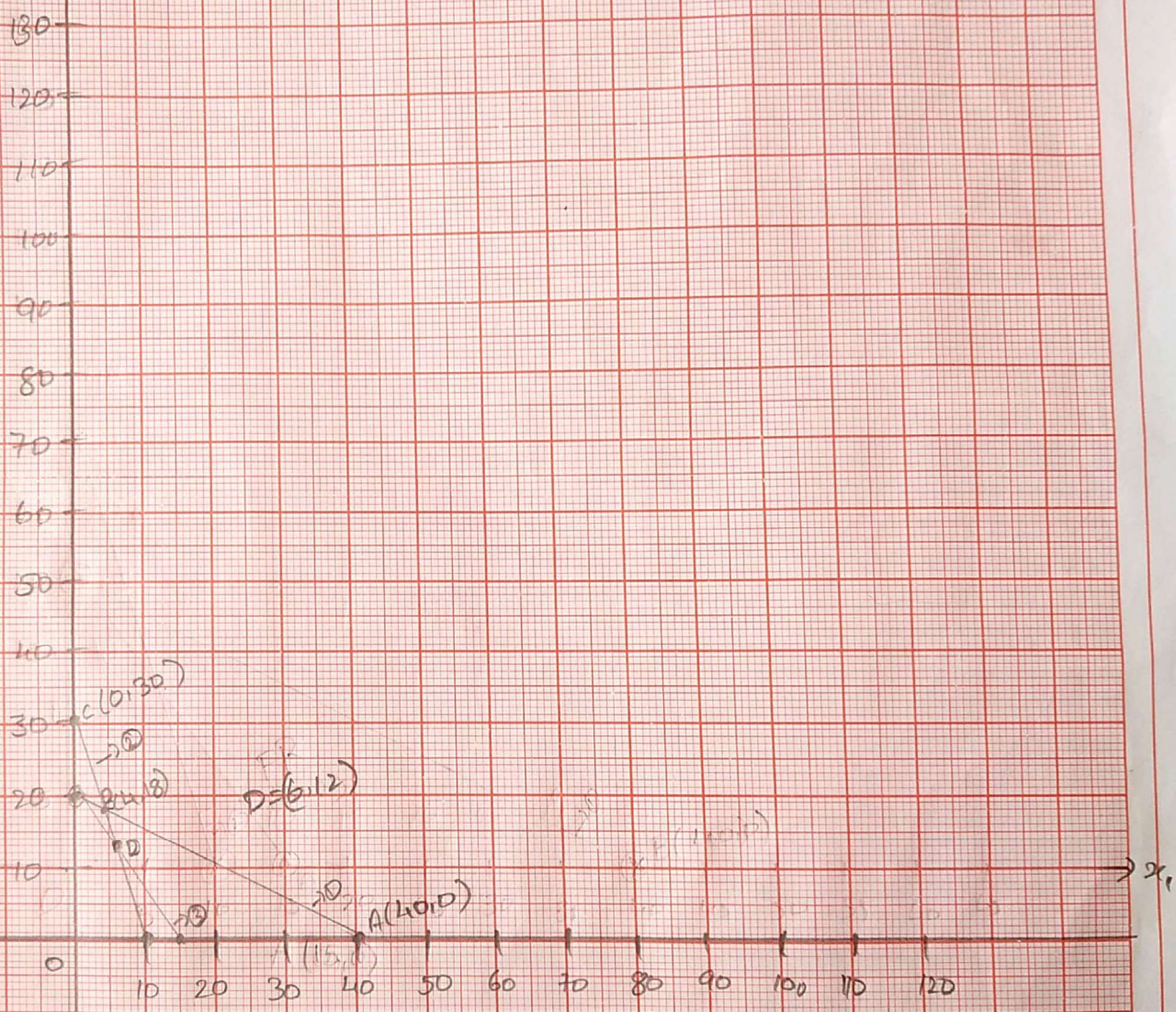
$$d = 120 + 120$$

$$d = 240$$

Scale:

x_1 1 cm = 10 units

x_2 1 cm = 10 units



4) Solve the following LPP by using Graphical method.

Objective function:

$$100x_1 + 60x_2$$

STC

$$5x_1 + 10x_2 \leq 50$$

$$8x_1 + 2x_2 \geq 16$$

$$3x_1 + 2x_2 \geq 6$$

NNC

$$x_1 \geq x_2 \quad x_1, x_2 \geq 0$$

Given

$$5x_1 + 10x_2 \leq 50 \rightarrow (1)$$

$$\text{Now } x_1 = 0$$

$$5(0) + 10x_2 = 50$$

$$0 + 10x_2 = 50$$

$$x_2 = \frac{50}{10}$$

$$x_2 = 5$$

x_1	0	10
x_2	5	0

(0,5) (10,0)

x_1	0	10
x_2	5	0

③ $\rightarrow$ 10 mark

$$0 = 5x_1 - 10x_2$$

$$0 = 10x_2$$

$$0 = 10x_2 - 50x_1$$

$$0 = 10x_2 - 50$$

$$10x_2 = 50$$

$$x_2 = 5$$

Now $x_2 = 0$

$5x_1 + 10x_2 = 50$

$5x_1 + 10(0) = 50$

$x_1 = \frac{50}{5}$

$x_1 = 10$

from eq $\rightarrow$ ②

$8x_1 + 2x_2 = 16$

Now $x_1 = 0$

$8(0) + 2x_2 = 16$

$0 + 2x_2 = 16$

$x_2 = \frac{16}{2}$

$x_2 = 8$

Now $x_2 = 0$

$8x_1 + 2(0) = 16$

$8x_1 + 0 = 16$

$x_1 = \frac{16}{8}$

$x_1 = 2$

from eq $\rightarrow$ ③

$3x_1 - 2x_2 = 6$

Now $x_1 = 0$

$3(0) - 2x_2 = 6$

$0 - 2x_2 = 6$

$-x_2 = \frac{6}{2}$

$x_2 = -3$

x_1	0	2
x_2	8	0
	(0, 8)	(2, 0)

x_1	0	2
x_2	-3	0

$$\text{Now } x_2 = 0$$

$$3x_1 - 2(0) = 6$$

$$3x_1 - 0 = 6$$

$$x_1 = \frac{6}{3}$$

$$\boxed{x_1 = 2}$$

From the above graph
ABC is a feasible region.

$$A(10,0) \quad B(4,3) \quad C(2,0)$$

$$= 100(10) + 60(0)$$

$$= 1,000 + 0$$

$$= 1,000$$

$$= 100(4) + 60(3)$$

$$= 400 + 180$$

$$= 580$$

$$= 100(2) + 60(0)$$

$$= 200 + 0$$

$$= 200$$

$\therefore$ optimization of 1000 at $x_1 = 10$ $x_2 = 0$

by using Graphical Method.

Vidhya

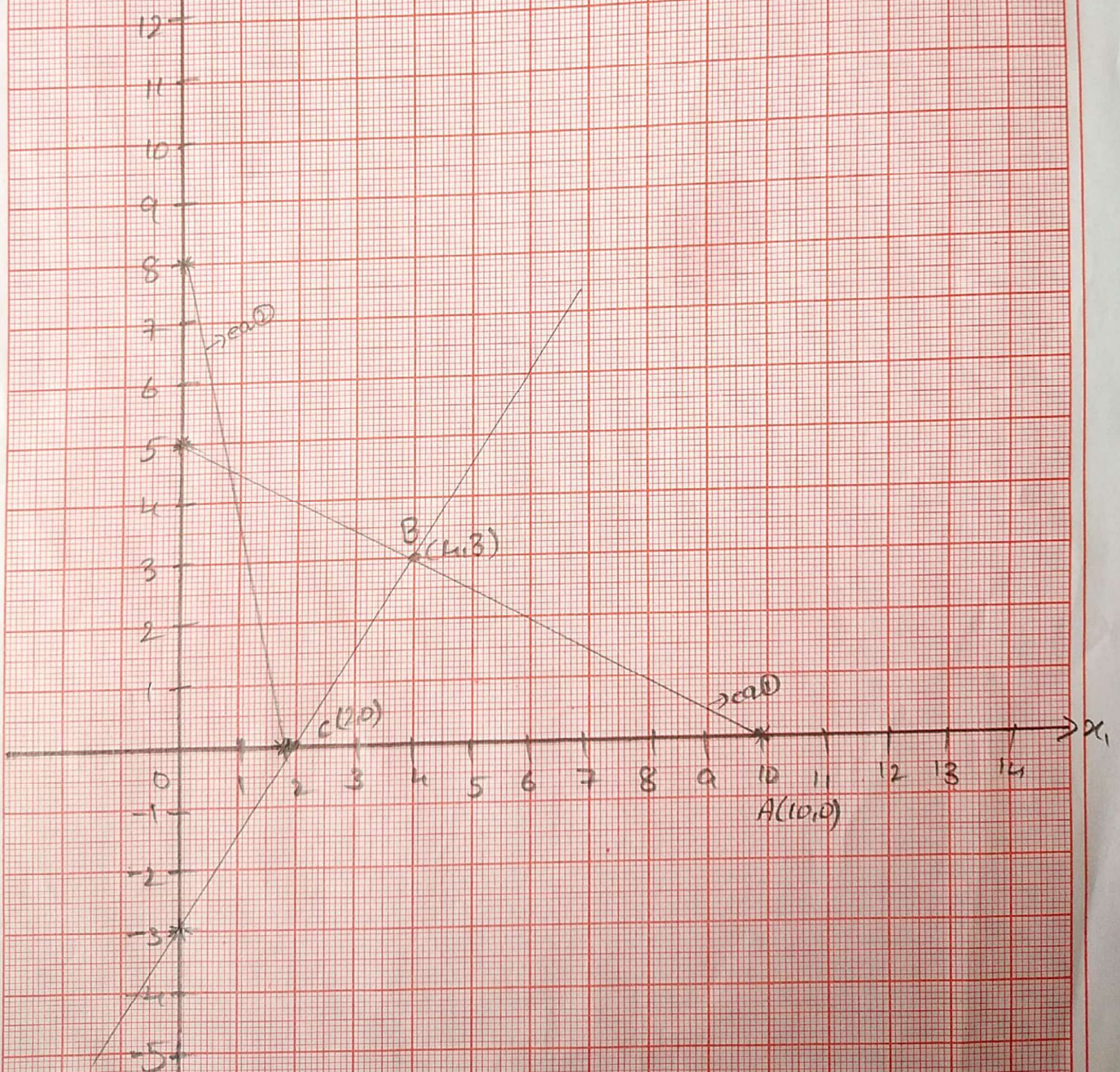
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Scale:

$x_1 = 1 \text{ cm} = 1 \text{ unit}$

$x_2 = 1 \text{ cm} = 1 \text{ unit}$



Simplex method problems

optimality conditions

for max:

$$\text{All } C_j - Z_j \leq 0 \quad [b, -V_c \text{ values}]$$

for min:

$$\text{All } C_j - Z_j \geq 0 \quad [b, +V_c \text{ values}]$$

Solving the LPP by using simplex method

Objective function:

$$\text{Max } Z = 12x_1 + 16x_2$$

STC

$$10x_1 + 20x_2 \leq 120$$

$$8x_1 + 8x_2 \leq 80$$

NNC

$$x_1, x_2 \geq 0$$

Sol Given problem is maximization type & given constraints are less than (or) equal to

→ So we introduce slack variables.

→ we add slack variables to convert inequalities.

$$\text{Max } Z = 12x_1 + 16x_2 + 0s_1 + 0s_2$$

STC

$$10x_1 + 20x_2 + s_1 = 120$$

$$8x_1 + 8x_2 + s_2 = 80$$

NNR

$$x_1, x_2, s_1, s_2 \geq 0$$

Initial Simplex table

co-efficient basic variables		key element		Key column		Key row (or) pivot row	
C_B	C_j	12	16	0	0	sol	ratio
	BV	x_1	x_2	S_1	S_2		
0	S_1	10	20	1	0	120	$\frac{120}{20} = 6$
0	S_2	8	8	0	1	80	$\frac{80}{8} = 10$
Z_j		0	0	0	0		
$S-Z_j$		12	16	0	0		

⇒ optimally condition is not reached in initial simplex table i.e., all $C_j - Z_j \neq 0$, so optimally is not reached the go to Iteration ①.

⇒ For Iteration 1.

⇒ S_1 is leaving variable.

⇒ x_2 is entering variable.

Z_j Values:

$$(0 \times 10) + (0 \times 8) = 0$$

$$(0 \times 20) + (0 \times 8) = 0$$

$$(0 \times 1) + (0 \times 0) = 0$$

$$(0 \times 0) + (0 \times 1) = 0$$

$$(0 \times 120) + (0 \times 80) = 0$$

$C_j - Z_j$ Values:

$$12 - 0 = 12$$

$$16 - 0 = 16$$

$$0 - 0 = 0$$

$$0 - 0 = 0$$

Ratio Formula:

$$\text{Ratio} = \frac{\text{Solution}}{\text{Key column Element}}$$

$$= \frac{120}{20} = 6$$

$$= \frac{80}{8} = 10$$

Here,

Iteration-I

B _i	C _j	12	16	0	0	RHS	Ratio
		x ₁	x ₂	s ₁	s ₂		
16	x ₂	$\frac{1}{2}$	1	$\frac{1}{20}$	0	6	12
0	s ₂	4	0	$-\frac{2}{5}$	1	32	8
2	s ₁	8	16	$\frac{4}{5}$	0	96	
5	s ₂	4	0	$-\frac{4}{5}$	0		

x₂ values

Key row
Key column element

$$= \frac{10}{20} = \frac{1}{2}$$

$$\frac{20}{20} = 1$$

$$\frac{1}{20} = \frac{1}{20}$$

$$\frac{0}{20} = 0$$

$$\frac{120}{20} = 6$$

s₂ values: [New values]

Formula:

$$\text{New value} = \text{Old value} - \frac{\text{CKCE} \times \text{CKRE}}{\text{KE}}$$

$$\therefore \text{NV} = \text{OV} - \frac{\text{CKCE} \times \text{CKRE}}{\text{KE}}$$

CKCE $\Rightarrow$ corresponding key column element

CKRE $\Rightarrow$ corresponding key Row element

KE $\Rightarrow$ key element

$$= 8 - \frac{8 \times 10}{20} = 4$$

$$= 8 - \frac{8 \times 20}{20} = 0$$

$$= 80 - \frac{8 \times 1}{20} = \frac{-2}{5}$$

$$= 01 - \frac{8 \times 0}{20} = 01$$

$$= 80 - \frac{8 \times 120}{20} = 32$$

Z_j Values:

$$(16 \times \frac{1}{2}) + (0 \times 4) = 8$$

$$(16 \times 1) + (0 \times 0) = 16$$

$$(16 \times \frac{1}{2} \times 0) + (0 \times \frac{-2}{5}) = \frac{4}{5}$$

$$(16 \times 0) + (0 \times 1) = 0$$

$$(16 \times 6) + (0 \times 32) = 96$$

C_j - Z_j Values:

$$12 - 8 = 4$$

$$16 - 16 = 0$$

$$0 - \frac{4}{5} = -\frac{4}{5}$$

$$0 - 0 = 0$$

Ratio: $\frac{6}{\frac{1}{2}} = \frac{6}{1} \times \frac{2}{1} = 12$

$$\frac{32}{4} = 8$$

So, optimally C_j - Z_j values are not equal, then go to iteration-2.

⇒ S₂ is leaving variable

⇒ x₁ is entering variable

Iteration-II

C _B i	C _j	12	16	0	0		
	BV	x ₁	x ₂	S ₁	S ₂	sol	Ratio
16	x ₂	0	1	$\frac{1}{10}$	$-\frac{1}{8}$	2	
12	x ₁	1	0	$-\frac{1}{10}$	$\frac{1}{4}$	8	
Z _j		12	16	$\frac{2}{5}$	1	128	
C _j - Z _j		0	0	$-\frac{2}{5}$	0		

Now C_j - Z_j values are equal to the maximization.

② Solving the LPP by using Simplex method

$$\text{Max } Z = 40x_1 + 35x_2$$

STC

$$2x_1 + 4x_2 \leq 60$$

$$3x_1 + 3x_2 \leq 96$$

NNR

$$x_1, x_2 \geq 0$$

sol Given problem is maximization type & given constraints are less than (or) equal to

⇒ So we introduce slack variables.

⇒ we add slack variables to convert inequalities into equalities.

$$\text{Max } Z = 40x_1 + 35x_2 + 0s_1 + 0s_2 \quad [0, F]$$

STC

$$2x_1 + 4x_2 + s_1 = 60$$

$$3x_1 + 3x_2 + s_2 = 96$$

NNR

$$x_1, x_2, s_1, s_2 \geq 0$$

initial simplex table

CB _i	C _j	40	35	0	0		
	BV	x_1	x_2	s_1	s_2	θ	ratio
0	s_1	2	4	1	0	60	30 ←
0	s_2	3	3	0	1	96	32
Z_j		0	0	0	0	0	
$C_j - Z_j$		40	35	0	0		

zj values

$$(0 \times 2) + (0 \times 3) = 0$$

$$(0 \times 4) + (0 \times 3) = 0$$

$$(0 \times 1) + (0 \times 0) = 0$$

$$(0 \times 0) + (0 \times 1) = 0$$

$$(0 \times 60) + (0 \times 96) = 0$$

Cj - zj values

$$40 - 0 = 40$$

$$35 - 0 = 35$$

$$0 - 0 = 0$$

$$0 - 0 = 0$$

Ratio: $\frac{801}{KCE}$

$$= \frac{60}{2} = 30$$

$$= \frac{96}{2} = 32$$

$\therefore$ Now S_1 is leaving the variable
 x_1 is entering the variable

Iteration - I

CB _i	C _j	40	35	0	0		
	BV	x_1	x_2	S_1	S_2		
40	x_1	1	2	$\frac{1}{2}$	0	30	
0	S_2	0	-3	$-\frac{3}{2}$	1	6	
	z_j	40	80	20	0		
	$C_j - z_j$	0	-45	-20	0		1200

S_2 values:

x_1 values:

$$\frac{2}{2} = 1$$

$$\frac{4}{2} = 2$$

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{0}{2} = 0$$

$$\frac{60}{2} = 30$$

Formula:

$$NV = OV - \frac{CKCE \times CKRE}{K \cdot E}$$

$$= 3 - \frac{3 \times 2}{2} = 0$$

$$= 3 - \frac{3 \times 4}{2} = -3$$

$$= 0 - \frac{3 \times 1}{2} = -\frac{3}{2}$$

$$= 1 - \frac{3 \times 0}{2} = 1$$

$$= 96 - \frac{3 \times 60}{2} = 6$$

z_j values:

$$(40 \times 1) + (0 \times 0) = 40$$

$$(40 \times 2) + (0 \times -3) = 80$$

$$(40 \times \frac{1}{2}) + (0 \times -\frac{3}{2}) = 20$$

$$(40 \times 0) + (0 \times 1) = 0$$

$$(40 \times 30) + (0 \times 6) = 1200$$

$C_j - Z_j$ Values:

$$40 - 40 = 0$$

$$35 - 80 = -45$$

$$0 - 20 = -20$$

$$0 - 0 = 0$$

Verification:

$$x_1 = 30, x_2 = 0$$

$$\max z = 40(30) + 35(0)$$

$$= 1200$$

∴ Hence it is proved.

3) Solve the LPP by using simplex method.

$$\max z = 6x_1 + 8x_2 \quad [O.F.]$$

STC

$$5x_1 + 10x_2 \leq 60$$

$$4x_1 + 4x_2 \leq 40$$

NNR

$$x_1, x_2 \geq 0$$

Sol Given problem is maximization type & given constraints are less than (or) equal to

⇒ so we introduce slack variables.

⇒ we add slack variables to convert inequalities to equalities.

$$\max z = 6x_1 + 8x_2 + 0s_1 + 0s_2 \quad [O.F.]$$

STC

$$5x_1 + 10x_2 + s_1 = 60$$

$$4x_1 + 4x_2 + s_2 = 40$$

NNR

$$x_1, x_2, s_1, s_2 \geq 0$$

initial simplex table

CB _i	C _j	6	8	0	0		
	BV	x ₁	x ₂	S ₁	S ₂	b _i	ratio
0	S ₁	5	10	1	0	60	6
0	S ₂	4	4	0	1	40	10
Z _j		0	0	0	0	0	
C _j - Z _j		6	8	0	0		

Z_j Values :

$$(0 \times 5) + (0 \times 4) = 0$$

$$(0 \times 10) + (0 \times 4) = 0$$

$$(0 \times 1) + (0 \times 0) = 0$$

$$(0 \times 0) + (0 \times 1) = 0$$

$$(0 \times 60) + (0 \times 40) = 0$$

C_j - Z_j Values

$$6 - 0 = 6$$

$$8 - 0 = 8$$

$$0 - 0 = 0$$

$$0 - 0 = 0$$

Ratio: $\frac{\text{Sol.}}{\text{KCE}}$

$$= \frac{60}{10} = 6$$

$$= \frac{40}{4} = 10$$

∴ Now optimally C_j - Z_j values are not equal.

Then go to iteration-I.

⇒ S₁ is leaving variable

⇒ x₂ is entering variable.

Iteration: I

CB _i	C _j	6	8	0	0		
	BV	x ₁	x ₂	S ₁	S ₂	b _i	ratio
8	x ₂	$\frac{1}{2}$	1	$\frac{1}{10}$	0	6	12
0	S ₂	2	0	$-\frac{2}{5}$	1	16	8
Z _j		4	8	$\frac{4}{5}$	0	48	
C _j - Z _j		2	0	$-\frac{4}{5}$	0		

x_2 Values:

$$\frac{5}{10} = \frac{1}{2}$$

$$\frac{10}{10} = 1$$

$$\frac{1}{10} = \frac{1}{10}$$

$$\frac{0}{10} = 0$$

$$\frac{60}{10} = 6$$

s_2 Values:

$$NU = 0.1 - \frac{CKCE \times CKRE}{K \cdot E}$$

		$0 = 4 - \frac{4 \times 5}{10} = 2$			
		$\frac{1}{10} = 4 - \frac{4 \times 10}{10} = 0$			
		$= 0 - \frac{4 \times 1}{10} = -\frac{2}{5}$			

Ratio: $\frac{sol}{K \cdot CE}$

$$= \frac{6}{\frac{1}{2}} = 12$$

$$= \frac{16}{2} = 8$$

$$= 1 - \frac{4 \times 0}{10} = 1$$

$$= 40 - \frac{4 \times 60}{10} = 16$$

Z_j Values:

$$(8 \times \frac{1}{2}) + (0 \times 2) = 4$$

$$(8 \times 1) + (0 \times 0) = 8$$

$$(8 \times \frac{1}{10}) + (0 \times -\frac{2}{5}) = \frac{4}{5}$$

$$(8 \times 0) + (0 \times 1) = 0$$

$$(8 \times 6) + (0 \times 16) = 48$$

$C_j - Z_j$ Values

$$6 - 4 = 2$$

$$8 - 8 = 0$$

$$0 - \frac{4}{5} = -\frac{4}{5}$$

$$0 - 0 = 0$$

$\therefore$ Now optimally, $C_j - Z_j$ values are not equal.

Then go to iteration-II

$\Rightarrow s_2$ is leaving variable

$\Rightarrow x_1$ is entering variable

Iteration = II

OB _i	C _j	6	8	0	0		
	BV	x ₁	x ₂	s ₁	s ₂	sol	Ratio
8	x ₂	0	1	$\frac{1}{5}$	$-\frac{1}{4}$	2	
6	x ₁	1	0	$-\frac{1}{5}$	2	8	
Z _j		6	8	$\frac{2}{5}$	10		
G-Z _j		0	0	$-\frac{2}{5}$	0	64	

x₁ values can be find out x₂ values

$$\frac{2}{2} = 1$$

$$\frac{0}{2} = 0$$

$$\frac{-2}{5} = -\frac{1}{5}$$

$$\frac{2}{1} = 2$$

$$\frac{16}{2} = 8$$

Z_j values:

$$(8 \times 0) + (6 \times 1) = 6$$

$$(8 \times 1) + (6 \times 0) = 8$$

$$(8 \times \frac{1}{5}) + (6 \times -\frac{1}{5}) = \frac{2}{5}$$

$$(8 \times -\frac{1}{4}) + (6 \times 2) = 10$$

$$(8 \times 2) + (6 \times 8) = 64$$

∴ The maximization G-Z_j values are to be a equal.

Verification:

$$\max Z = 6x_1 + 8x_2$$

$$= 6(8) + 8(2)$$

$$= 48 + 16$$

$$= 64 \Rightarrow \text{Hence it is proved.}$$

4) Solve the LPP by using simplex method. 2 constraints

$$\text{Max } Z = 10x_1 + 15x_2 + 20x_3$$

STC

$$2x_1 + 4x_2 + 6x_3 \leq 24$$

$$3x_1 + 9x_2 + 6x_3 \leq 30$$

NNR

$$x_1, x_2, x_3 \geq 0$$

∴ Given problem is maximization type & given constraints are less than (or) equal to

⇒ so we introduce slack variables

⇒ we add slack variables to convert inequalities into equalities.

$$\text{Max } Z = 10x_1 + 15x_2 + 20x_3 + 0s_1 + 0s_2 \text{ [OF]}$$

STC: Max $Z = 10x_1 + 15x_2 + 20x_3 + 0s_1 + 0s_2 \text{ [OF]}$

2

STC

$$2x_1 + 4x_2 + 6x_3 + s_1 = 24$$

$$3x_1 + 9x_2 + 6x_3 + s_2 = 30$$

NNR:

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

initial simplex table

CB _i	C _j	10	15	20	0	0		
	BU	x_1	x_2	x_3	s_1	s_2	M	ratio
0	s_1	2	4	6	1	0	24	4
0	s_2	3	9	6	0	1	30	5
Z_j		0	0	0	0	0	0	
$C_j - Z_j$		10	15	20	0	0		

Zj values:

$$(0 \times 2) + (0 \times 3) = 0$$

$$(0 \times 4) + (0 \times 9) = 0$$

$$(0 \times 6) + (0 \times 6) = 0$$

$$(0 \times 1) + (0 \times 0) = 0$$

$$(0 \times 0) + (0 \times 1) = 0$$

$$(0 \times 24) + (0 \times 30) = 0$$

ratio
Solution
KCE

$$= \frac{24}{6} = 4$$

$$= \frac{30}{6} = 5$$

∴ The optimally $C_j - Z_j$ is not causal, then go to the iteration-I table.

⇒ Now S_1 is leaving variable

⇒ x_3 is entering variable

Iteration-I

CBi	Cj	10	15	20	0	0	sol	ratio
	BV	x_1	x_2	x_3	S_1	S_2		
20	x_3	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{6}$	0	4	12
0	S_2	①	5	0	-1	1	6	6 ←
Z_j		$\frac{20}{3}$	$\frac{40}{3}$	20	$\frac{10}{3}$	0	80	
$C_j - Z_j$		$\frac{10}{3}$	$\frac{5}{3}$	0	$-\frac{10}{3}$	0		

x_3 values:

$$\frac{2}{6} = \frac{1}{3}$$

$$\frac{4}{6} = \frac{2}{3}$$

$$\frac{6}{6} = 1$$

$$\frac{1}{6} = \frac{1}{6}$$

$$\frac{0}{6} = 0$$

$$\frac{24}{6} = 4$$

CBi	Cj	10	15	20	0	0	sol	ratio
	BV	x_1	x_2	x_3	S_1	S_2		
20	x_3	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{6}$	0	4	12
0	S_2	①	5	0	-1	1	6	6 ←
Z_j		$\frac{20}{3}$	$\frac{40}{3}$	20	$\frac{10}{3}$	0	80	
$C_j - Z_j$		$\frac{10}{3}$	$\frac{5}{3}$	0	$-\frac{10}{3}$	0		

S₂ Values:

$$NV = old - \frac{CKCE \times CKRE}{KE}$$

$$= 3 - \frac{2 \times 6}{6} = 1$$

$$= 9 - \frac{4 \times 6}{6} = 5$$

$$= 6 - \frac{6 \times 6}{6} = 0$$

$$= 0 - \frac{1 \times 6}{6} = -1$$

$$= 1 - \frac{0 \times 6}{6} = 1$$

$$= 30 - \frac{24 \times 6}{6} = 6$$

Z_j Values:

$$(20 \times \frac{1}{3}) + (0 \times 1) = \frac{20}{3}$$

$$(20 \times \frac{2}{3}) + (0 \times 5) = \frac{40}{3}$$

$$(20 \times 1) + (0 \times 0) = 20$$

$$(20 \times \frac{1}{6}) + (0 \times -1) = \frac{10}{3}$$

$$(20 \times 0) + (0 \times 1) = 0$$

$$(20 \times 4) + (0 \times 6) = 80$$

C_j - Z_j Values:

$$10 - \frac{20}{3} = \frac{10}{3}$$

$$15 - \frac{40}{3} = \frac{5}{3}$$

$$20 - 20 = 0$$

$$0 - \frac{10}{3} = -\frac{10}{3}$$

$$0 - 0 = 0$$

Ratio: $\frac{sol}{KCE}$

$$= \frac{4}{\frac{1}{3}} = \frac{4}{1} \times \frac{3}{1} = 12$$

$$= \frac{6}{1} = 6$$

∴ optimality is C_j - Z_j values are not equal then go to iteration (2)

Now, S_2 is leaving variable
 x_1 is entering variable

iteration: II

CB_i	C_j	10	15	20	0	0		
	B_v	x_1	x_2	x_3	S_1	S_2	RHS	ratio
20	x_3	0	-1	1	$\frac{1}{2}$	$-\frac{1}{3}$	2	
10	x_1	1	5	0	-1	1	6	
Z_j		10	30	20	0	$\frac{10}{3}$	100	
$C_j - Z_j$		0	-15	0	0	$-\frac{10}{3}$		

x_1 values:

$$\frac{1}{1} = 1$$

$$\frac{5}{1} = 5$$

$$\frac{-1}{1} = -1$$

$$\frac{0}{1} = 0$$

$$\frac{1}{1} = 1$$

$$\frac{6}{1} = 6$$

x_3 values

$$NV = OV - \frac{CKCE \times CKRE}{K \cdot E}$$

$$= \frac{1}{3} - \frac{\frac{1}{3} \times 1}{1} = 0$$

$$= \frac{2}{3} - \frac{\frac{1}{3} \times 5}{1} = -1$$

$$= 1 - \frac{\frac{1}{3} \times 6}{1} = 1$$

$$= \frac{1}{6} - \frac{\frac{1}{3} \times -1}{1} = \frac{1}{2}$$

$$= 0 - \frac{\frac{1}{3} \times 1}{1} = -\frac{1}{3}$$

$$= 4 - \frac{\frac{1}{3} \times 6}{1} = 2$$

$C_j - 2_j$ values:

$$10 - 10 = 0$$

$$15 - 30 = -15$$

$$20 - 20 = 0$$

$$0 - 0 = 0$$

$$0 - \frac{10}{3} = -\frac{10}{3}$$

Verification

$$= 10x_1 + 15x_2 + 20x_3$$

$$= 10(6) + 15(0) + 20(2)$$

$$= 60 + 0 + 40$$

$$= 100 //$$

$\therefore$ Hence it is proved.

Hence the optimality condition is true (or) equal.

5) Solve the Lpp by using simplex method with 3 variables & three (3) constraints.

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3 \quad [O.F.]$$

STC:

$$2x_1 + 3x_2 \leq 8$$

$$2x_2 + 5x_3 \leq 10$$

$$3x_1 + 2x_2 + 4x_3 \leq 15$$

NMR

$$x_1, x_2, s_1, s_2 \geq 0$$

sol Given problem it is maximization type & given constraints are less than (or) equal to.

→ So we can introduce slack variables.

→ We add slack variables to convert inequalities into equalities

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3 \quad [\text{OF}]$$

STC

$$2x_1 + 3x_2 + s_1 = 8$$

$$2x_2 + 5x_3 + s_2 = 10$$

$$3x_1 + 2x_2 + 4x_3 + s_3 = 15$$

NNR

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Initial Simplex method

CB _i	C _j	3	5	4	0	0	0		
	BU	x_1	x_2	x_3	s_1	s_2	s_3	80	Ratio
0	s_1	2	3	0	1	0	0	8	$\frac{8}{3} = 2.66$
0	s_2	0	2	5	0	1	0	10	5
0	s_3	3	2	4	0	0	1	15	$\frac{15}{2} = 7.5$
0	Z	0	0	0	0	0	0		
0	$Z - C_j$	3	5	4	0	0	0		

$$\text{Ratio} = \frac{8}{3} = \frac{8}{3}$$

$$= \frac{10}{2} = 5$$

$$= \frac{15}{2} = \frac{15}{2}$$

∴ optimum condition is not equal. then go to iteration-I

→ So s_1 is leaving variable
→ x_2 is entering "

Iteration - I

CB_i	C_j	3	5	4	0	0	0		
	BV	x_1	x_2	x_3	s_1	s_2	s_3	b_i	ratio
5	x_1	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{8}{3}$	-
0	s_2	$-\frac{4}{3}$	0	5	$-\frac{2}{3}$	1	0	$\frac{14}{3}$	$\frac{14}{15} = (0.933)$
0	s_3	$\frac{5}{3}$	0	4	$-\frac{2}{3}$	0	1	$\frac{29}{3}$	$\frac{29}{12} = (2.41)$
	x_3	$\frac{10}{3}$	5	0	$\frac{5}{3}$	0	0	$\frac{40}{3}$	
	s_1	$-\frac{1}{3}$	0	4	$-\frac{5}{3}$	0	0		

x_2 Value

$$\begin{aligned}\frac{2}{3} &= \frac{2}{3} \\ \frac{3}{3} &= 1 \\ \frac{0}{3} &= 0 \\ \frac{1}{3} &= \frac{1}{3} \\ \frac{0}{3} &= 0 \\ \frac{0}{3} &= 0 \\ \frac{8}{3} &= \frac{8}{3}\end{aligned}$$

s_2 Value

$$NV = OV - \frac{CKEXCKRE}{KE}$$

$$= 0 - \frac{2 \times 2}{3} = -\frac{4}{3}$$

$$= 2 - \frac{2 \times 3}{3} = 0$$

$$= 5 - \frac{2 \times 0}{3} = 5$$

$$= 0 - \frac{2 \times 1}{3} = -\frac{2}{3}$$

$$= 1 - \frac{2 \times 0}{3} = 1$$

$$= 0 - \frac{2 \times 0}{3} = 0$$

$$= 10 - \frac{2 \times 8}{3} = \frac{14}{3}$$

s_3 Value

$$NV = OV - \frac{CKEXCKRE}{KE}$$

$$= 3 - \frac{2 \times 2}{3} = \frac{5}{3}$$

$$= 2 - \frac{2 \times 3}{3} = 0$$

$$= 4 - \frac{2 \times 0}{3} = 4$$

$$= 0 - \frac{2 \times 1}{3} = -\frac{2}{3}$$

$$= 0 - \frac{2 \times 0}{3} = 0$$

$$= 1 - \frac{2 \times 0}{3} = 1$$

$$= 15 - \frac{2 \times 8}{3} = \frac{29}{3}$$

Ratio $\frac{8}{3} = 0(\infty)$

$$\frac{14}{3} = \frac{14}{15} (0.93)$$

$$\frac{29}{3} = \frac{29}{12} (2.41)$$

optimum is not equal to 0 (or) +ve value
then go to iteration-II.

→ so s_2 is leaving variable

→ x_3 is entering variable

Iteration-III

CB _i	C _i	3	5	4	0	0	0		
	BV	x_1	x_2	x_3	s_1	s_2	s_3	RHS	ratio
5	x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{8}{3}$	4
4	x_3	$-\frac{4}{15}$	0	1	$-\frac{2}{15}$	$\frac{1}{5}$	0	$\frac{14}{15}$	$\frac{7}{2}$
0	s_3	$\frac{41}{15}$	0	0	$-\frac{2}{15}$	$-\frac{4}{5}$	1	$\frac{89}{15}$	$\frac{89}{41} = 2.17 \leftarrow$
ΣC_j		$\frac{34}{15}$	5	4	$-\frac{17}{15}$	$-\frac{4}{5}$	0	$\frac{259}{15}$	
$\Sigma C_j - Z_j$		$\frac{11}{15} \uparrow$	0	0	$-\frac{17}{15}$	$-\frac{4}{5}$	0		

x_3 = value)

$$-\frac{4}{3} \times \frac{1}{5} = -\frac{4}{15}$$

$$\frac{0}{5} = 0$$

$$\frac{5}{5} = 1$$

$$-\frac{2}{3} \times \frac{1}{5} = -\frac{2}{15}$$

$$\frac{1}{5} = \frac{1}{5}$$

$$\frac{0}{5} = 0$$

$$\frac{14}{3} \times \frac{1}{5} = \frac{14}{15}$$

x_2 value)

$$NV = OV - \frac{CK \times CKRE}{KE}$$

$$= \frac{2}{3} - \frac{0 \times -\frac{4}{3}}{5} = \frac{2}{3}$$

$$= 1 - \frac{0 \times 0}{5} = 1$$

$$= 0 - \frac{0 \times 5}{5} = 0$$

$$= \frac{1}{3} - \frac{0 \times -\frac{2}{3}}{5} = \frac{1}{3}$$

$$= 0 - \frac{0 \times 1}{5} = 0$$

$$= 0 - \frac{0 \times 0}{5} = 0$$

$$= \frac{8}{3} - \frac{0 \times \frac{14}{3}}{5} = \frac{8}{3}$$

S₃ Value

$$NV = OV - \frac{CKRE \times CKRE}{KE}$$

$$= \frac{5}{3} - \frac{4 \times \frac{-4}{3}}{5} = \frac{41}{15}$$

$$= 0 - \frac{4 \times 0}{5} = 0$$

$$= 4 - \frac{4 \times 5}{5} = 0$$

$$= \frac{-2}{3} - \frac{4 \times \frac{-2}{3}}{5} = \frac{-2}{15}$$

$$= 0 - \frac{4 \times 1}{5} = \frac{-4}{5}$$

$$= 1 - \frac{4 \times 0}{5} = 1$$

$$= \frac{29}{3} - \frac{4 \times \frac{14}{3}}{5} = \frac{89}{15}$$

∴ optimum is not equal to '0' (or) the values then go to iteration - III

→ so S₃ is leaving variable

→ x₁ is entering " iteration - III

CB _i	C _i	3	5	4	0	0	0		
	BV	x ₁	x ₂	x ₃	s ₁	s ₂	s ₃	h ₀	ratio
5	x ₂	0	1	0	$\frac{15}{41}$	$\frac{8}{41}$	$\frac{-10}{41}$	$\frac{50}{41}$	1.219
4	x ₃	0	0	$\frac{6}{41}$	$\frac{8-6}{41}$	$\frac{4-5}{41}$	$\frac{62}{41}$	$\frac{62}{41}$	1.512
3	x ₁	1	0	0	$\frac{-2}{41}$	$\frac{-12}{41}$	$\frac{15}{41}$	$\frac{89}{41}$	
2	3	5	4	$\frac{45}{41}$	$\frac{24}{41}$	$\frac{11}{41}$	$\frac{765}{41}$	$\frac{765}{41}$	18.6585
5	2	0	0	0	$\frac{-45}{41}$	$\frac{-24}{41}$	$\frac{-11}{41}$		

x_1 value

$$\frac{41}{15} = 1$$

$$\frac{0}{41} = 0$$

$$\frac{0}{41} = 0$$

$$\frac{-2}{15} = \frac{-2}{41}$$

$$\frac{-4}{5} = \frac{-12}{41}$$

$$\frac{1}{41} = \frac{15}{41}$$

$$\frac{89}{15} = \frac{89}{41}$$

x_3 value

$$= -\frac{4}{15} - \frac{\frac{4}{15} \times \frac{41}{15}}{\frac{41}{15}} = 0$$

$$= 0 - \frac{-\frac{4}{15} \times 0}{\frac{41}{15}} = 0$$

$$= -\frac{2}{15} - \frac{-\frac{4}{15} \times \frac{-2}{15}}{\frac{41}{15}} = \frac{-6}{41}$$

$$= \frac{1}{5} - \frac{-\frac{4}{15} \times \frac{-4}{5}}{\frac{41}{15}} = \frac{5}{41}$$

$$= 0 - \frac{-\frac{4}{15} \times 1}{\frac{41}{15}} = \frac{4}{41}$$

$$= \frac{14}{15} - \frac{-\frac{4}{15} \times \frac{89}{15}}{\frac{41}{15}} = \frac{62}{41} = 1.512$$

x_2 value

$$= \frac{2}{3} - \frac{\frac{2}{3} \times \frac{41}{15}}{\frac{41}{15}} = 0$$

$$= 1 - \frac{\frac{2}{3} \times 0}{\frac{41}{15}} = 1$$

$$= 0 - \frac{\frac{2}{3} \times 0}{\frac{41}{15}} = 0$$

$$= \frac{1}{3} - \frac{\frac{2}{3} \times \frac{-2}{15}}{\frac{41}{15}} = \frac{15}{41}$$

$$= 0 - \frac{\frac{2}{3} \times \frac{-4}{5}}{\frac{41}{15}} = \frac{8}{41}$$

$$= 0 - \frac{\frac{2}{3} \times 1}{\frac{41}{15}} = \frac{-10}{41}$$

$$= \frac{8}{3} - \frac{\frac{2}{3} \times \frac{89}{15}}{\frac{41}{15}} = \frac{50}{41} = 1.219$$

$\therefore$ optimization is equal to in iteration-III,
maximization $x_1 = 2.1707, x_2 = 1.219, x_3 = 1.512,$

maximization $z = 18.65$

Hence it is proved.

1) solve the LPP by using simplex method

$$\min z = x_1 - 3x_2 + 2x_3 \quad [O.F.]$$

STC

$$3x_1 - x_2 + 2x_3 \leq 7$$

$$-2x_1 + 4x_2 \leq 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$

NNR

$$x_1, x_2, x_3 \geq 0$$

2) Given problem is minimization problem.

→ we can minimization problem into maximization problem with multiple of (-1) to the objective function. Now the (O.F) $\min z = \max [-z]$

$$\max z = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$$

STC

$$3x_1 - x_2 + 2x_3 + s_1 = 7$$

$$-2x_1 + 4x_2 + s_2 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$

NNR

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Simple initial table

CB _i	C _j	-1	3	-2	0	0	0		
	BV	x_1	x_2	x_3	s_1	s_2	s_3	sol	ratio
0	s_1	3	-1	2	1	0	0	7	-7
0	s_2	-2	4	0	0	1	0	12	3 ←
0	s_3	-4	3	8	0	0	1	10	$\frac{10}{3} = 3.33$
z_j		0	0	0	0	0	0	0	
$C_j - z_j$		-1	3 ↑	-2	0	0	0		

z_j values

$$(0 \times 3) + (0 \times 1 - 2) + (0 \times -4)$$

$$(0 \times -1) + (0 \times 4) + (0 \times 3)$$

$C_j - z_j$

$$-1 - 0 = -1$$

$$3 - 3 = 0$$

$$-2 - 0 = -2$$

Ratio: $\frac{\text{RHS}}{\text{KCE}} = \frac{7}{-1} = -7$

$$= \frac{12}{4} = 3$$

$$= \frac{10}{3} = \frac{10}{3} = 3.33$$

The maximization of optimally condition is not equal.

→ So S_2 is leaving variable

→ x_2 is entering variable

→ Then go to iteration-I

Iteration-I

	C_j	-1	3	-2	0	0	0	
C_B	B_V	x_1	x_2	x_3	S_1	S_2	S_3	Ratio
0	S_1	$\frac{5}{2}$	0	2	1	$\frac{1}{4}$	0	10/4 ←
3	x_1	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	3/6
0	S_3	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	1/5 (0.4)
z_j		-3/2	3	0	0	$\frac{3}{4}$	0	9
$C_j - z_j$		$\frac{1}{2}$ ↑	0	-2	0	$-\frac{3}{4}$	0	

(0.5)

x_2 values

$$\frac{-2}{4} = -\frac{1}{2}$$

$$\frac{4}{4} = 1$$

$$\frac{0}{4} = 0$$

$$\frac{0}{4} = 0$$

$$\frac{1}{4} = \frac{1}{4}$$

$$\frac{0}{4} = 0$$

$$\frac{12}{4} = 3$$

S₁ Values:

$$NV = OV - \frac{CKEXCKRE}{KE}$$

$$= 3 - \frac{-1 \times -2}{4} = \frac{5}{2}$$

$$= -1 - \frac{-1 \times 4}{4} = 0$$

$$= 2 - \frac{-1 \times 0}{4} = 2$$

$$= 1 - \frac{-1 \times 0}{4} = 1$$

$$= 0 - \frac{-1 \times 1}{4} = \frac{1}{4}$$

$$= 0 - \frac{-1 \times 0}{4} = 0$$

$$= 7 - \frac{-1 \times 12}{4} = 10$$

S₃ Values

$$NV = OV - \frac{CKEXCKRE}{KE}$$

$$= -4 - \frac{3 \times -2}{4} = -\frac{5}{2}$$

$$= 3 - \frac{3 \times 4}{4} = 0$$

$$= 8 - \frac{3 \times 0}{4} = 8$$

$$= 0 - \frac{3 \times 0}{4} = 0$$

$$= 0 - \frac{3 \times 1}{4} = -\frac{3}{4}$$

$$= 1 - \frac{3 \times 0}{4} = 1$$

$$= 10 - \frac{3 \times 12}{4} = 1$$

Ratio:

$$\frac{10}{\frac{5}{2}} = 4$$

$$\frac{3}{-\frac{1}{2}} = -6$$

$$\frac{1}{-\frac{5}{2}} = -\frac{2}{5} (0.4)$$

∴ optimal condition is not equal. Then we

go to iteration II.

⇒ so S₁ is leaving variable

⇒ x₁ is entering variable

Iteration-II

CB _i	C _j	1	3	-2	0	0	0	ratio
	BV	x ₁	x ₂	x ₃	s ₁	s ₂	s ₃	
-1	x ₁	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	4
3	x ₂	0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0	5
0	s ₃	0	0	10	1	$-\frac{1}{2}$	1	11
	z _j	-1	3	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	11
	z _j -c _j	0	0	$-\frac{12}{5}$	$-\frac{1}{5}$	$-\frac{4}{5}$	0	

x₁ values

$$\frac{\frac{5}{2}}{\frac{5}{2}} = 1$$

$$\frac{0}{\frac{5}{2}} = 0$$

$$\frac{\frac{2}{5}}{\frac{5}{2}} = \frac{4}{5} = 0.8$$

$$\frac{\frac{1}{5}}{\frac{5}{2}} = \frac{2}{5} = 0.4$$

$$\frac{\frac{1}{4}}{\frac{5}{2}} = \frac{1}{10} = 0.1$$

$$\frac{0}{\frac{5}{2}} = 0$$

$$\frac{10}{\frac{5}{2}} = 4$$

x₂ values

$$NU = 01 - \frac{CK \cdot EX \cdot CK \cdot RE}{KE}$$

$$= -\frac{1}{2} - \frac{-\frac{1}{2} \times \frac{5}{2}}{\frac{5}{2}} = 0$$

$$= 1 - \frac{-\frac{1}{2} \times 0}{\frac{5}{2}} = 1$$

$$= 0 - \frac{-\frac{1}{2} \times 2}{\frac{5}{2}} = \frac{2}{5} = (0.4)$$

$$= 0 - \frac{-\frac{1}{2} \times 1}{\frac{5}{2}} = \frac{1}{5}$$

$$= \frac{1}{4} - \frac{-\frac{1}{2} \times \frac{1}{4}}{\frac{5}{2}} = \frac{3}{10} = (0.3)$$

$$= 0 - \frac{-\frac{1}{2} \times 0}{\frac{5}{2}} = 0$$

$$= 3 - \frac{-\frac{1}{2} \times 10}{\frac{5}{2}} = 5$$

53 Values

$$NV = OV - \frac{CKE \times CRE}{k \cdot E}$$

$$= -\frac{5}{2} - \frac{-\frac{5}{2} \times \frac{5}{2}}{\frac{5}{2}} = 0$$

$$= 0 - \frac{-\frac{5}{2} \times 0}{\frac{5}{2}} = 0$$

$$= 8 - \frac{-\frac{5}{2} \times 2}{\frac{5}{2}} = 10$$

$$= 0 - \frac{-\frac{5}{2} \times 1}{\frac{5}{2}} = 1$$

$$= 1 - \frac{-\frac{5}{2} \times 0}{\frac{5}{2}} = 1$$

$$= 1 - \frac{-\frac{5}{2} \times 10}{\frac{5}{2}} = 11$$

∴ so optimum condition is reached in the iteration III. $x_1 = 4, x_2 = 5, x_3 = 0$ $z(\text{optimum}) = 11$

In minimization method $x_1 = -4, x_2 = -5, x_3 = 0$

$$z(\text{optimum}) = -11$$

∴ Hence it is proved.

Duality in Simplex

Formulation of Dual problem:

Primal to dual conversion:

Step: 1

→ First we convert the problem in canonical form.

$$\min \rightarrow \min \geq$$

$$\max \rightarrow \max \leq$$

→ change the objective function of maximisation in the primal into maximisation in the dual & vice-versa.

$$\max Z \rightarrow \min Z \geq$$

$$\min Z \rightarrow \max Z \leq$$

→ The no. of variable in the primal will be the no. of constraints in dual & vice versa.

$$\text{Primal} \begin{cases} \text{no. of variable} = n \\ \text{no. of constraints} = m \end{cases}$$

$$\text{dual} \begin{cases} \text{no. of constraints} = n \\ \text{no. of variables} = m \end{cases}$$

→ cost coefficient in objective function of the primal will be right hand side constants of the constraints in the dual & vice-versa.

→ For formulating constraints we considered the transpose of matrix.

$$\textcircled{1} \text{ Max } z = x_1 + 2x_2 + x_3$$

STC

$$2x_1 + x_2 - x_3 \leq 2$$

$$-2x_1 + x_2 - 5x_3 \geq -6$$

$$4x_1 + x_2 + x_3 \leq 6$$

NNR

$$x_1, x_2, x_3 \geq 0$$

Q2 Given

The cononical form

$$\text{Max } z = x_1 + 2x_2 + x_3 \text{ [Objective function]}$$

STC

$$2x_1 + x_2 - x_3 \leq 2$$

eq(2) is multiple by (-1). because this equation is in greater than form, then we will convert the maximization form.

$$2x_1 - x_2 + 5x_3 \leq 6$$

$$4x_1 + x_2 + x_3 \leq 6$$

max $\leq$
min $\geq$

NNR:

$$x_1, x_2, x_3 \geq 0$$

Primal	cononical	dual
min	min $\geq$	max $\leq$
max	max $\leq$	min $\geq$

$\Rightarrow$ Dual form y_1, y_2, y_3 be a dual variable

$$\text{min } z = 2y_1 + 6y_2 + 6y_3 \text{ [Objective function]}$$

change row to the column.

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 5 \\ 4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 4 \\ 1 & -1 & 1 \\ -1 & 5 & 1 \end{pmatrix}$$

Then STC

$$2y_1 + 2y_2 + 4y_3 \geq 1$$

$$y_1 - y_2 + y_3 \geq 2$$

$$-y_1 + y_2 + y_3 \geq 1$$

∴ When these constraints are converted into the minimization form with the coefficient of the objective function. [max] [like 1, 2, 1]

NNR: $y_1, y_2, y_3 \geq 0$

This is the formulation.

② $\max z = x_1 - x_2 + 3x_3$ [Objective function]

STC

$$x_1 + x_2 + x_3 \leq 10$$

$$2x_1 - x_3 \leq 2$$

$$2x_1 - 2x_2 + 3x_3 \leq 6$$

NNR: $x_1, x_2, x_3 \geq 0$

Q1 Given canonical form

dual form y_1, y_2, y_3 be a dual variable

$$\min z = 10y_1 + 2y_2 + 6y_3$$

STC

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 2 & -2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & -2 \\ 1 & -1 & 3 \end{bmatrix}$$

$$y_1 + 2y_2 + 2y_3 \geq 1$$

$$-y_1 - 2y_3 \geq -1$$

$$-y_1 - y_2 + 3y_3 \geq 3$$

$$\text{NNR: } y_1, y_2, y_3 \geq 0$$

Applications of duality:

1) Solve the following LPP using the result of its dual problem.

$$\min z = 24x_1 + 30x_2 \quad [\text{Objective function}]$$

STC

$$2x_1 + 3x_2 \geq 10$$

$$4x_1 + 9x_2 \geq 15$$

$$6x_1 + 6x_2 \geq 20$$

NNR

$$x_1, x_2 \geq 0$$

Sol Let cononical form

Dual form

Let y_1, y_2, y_3 are the dual variables

$$\max z = 10y_1 + 15y_2 + 20y_3$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 9 \\ 6 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 9 & 6 \end{bmatrix}$$

$$2y_1 + 4y_2 + 6y_3 \leq 24$$

$$3y_1 + 9y_2 + 6y_3 \leq 30$$

$$\text{NNR: } y_1, y_2, y_3 \geq 0$$

Canonical form duality

max $Z = 10y_1 + 15y_2 + 20y_3 + 0s_1 + 0s_2$ [O.P.]

STC

$$2y_1 + 4y_2 + 6y_3 + s_1 = 24$$

$$3y_1 + 9y_2 + 6y_3 + s_2 = 30$$

NMR

$$y_1, y_2, y_3, s_1, s_2 \geq 0$$

Simple iteration table

CB _i	C _j	10	15	20	0	0		
	BV	y_1	y_2	y_3	s_1	s_2	R _i	ratio
0	s_1	2	4	6	1	0	24	4 ←
0	s_2	3	9	6	0	1	30	5
	z_j	0	0	0	0	0	0	
	$C_j - z_j$	10	15	20	0	0		

$$\text{ratio} = \frac{RDI}{KCF} = \frac{24}{6} = 4$$

$$= \frac{30}{6} = 5$$

∴ s_1 is leaving variable
 y_3 is entering variable.
 then go to iteration-I

CB _i	C _j	10	15	20	0	0		
	BV	y_1	y_2	y_3	s_1	s_2	R _i	ratio
20	y_3	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{6}$	0	4	12
0	s_2	5	0	-1	1	1	6	6 ←
	z_j	$\frac{10}{3}$	$\frac{40}{3}$	20	$\frac{10}{3}$	0	80	
	$C_j - z_j$	$\frac{20}{3}$	$\frac{10}{3}$	0	$-\frac{10}{3}$	0		

y₃ values

$$\frac{2}{6} = \frac{1}{3}$$

$$\frac{4}{6} = \frac{2}{3}$$

$$\frac{6}{6} = 1$$

$$\frac{1}{6} = \frac{1}{6}$$

$$\frac{0}{6} = 0$$

$$\frac{24}{6} = 4$$

s₂ Value

$$OV - \frac{CKCE \times CKRE}{KE}$$

$$= 3 - \frac{6 \times 2}{6} = 1$$

$$= 9 - \frac{6 \times 4}{6} = 5$$

$$= 6 - \frac{6 \times 6}{6} = 0$$

$$= 0 - \frac{6 \times 1}{6} = -1$$

$$= 1 - \frac{6 \times 0}{6} = 1$$

$$= 30 - \frac{6 \times 24}{6} = 6$$

∴ optimization is not equal, then go to iteration-II

So, s₂ is leaving variable

y₁ is entering variable

Iteration-II

Bi	C _j	10	15	20	0	0		
	BV	y ₁	y ₂	y ₃	s ₁	s ₂	W	ratio
20	y ₃	0	-1	1	$\frac{1}{2}$	$-\frac{1}{3}$	2	
10	y ₁	1	5	0	-1	1	6	
10	y ₂	0	30	20	0	$\frac{10}{3}$	100	
4-21		0	-15	0	0	$-\frac{10}{3}$		

∴ optimization is in equal condition. All y₃ values are equal.

y₁ values

$$\frac{1}{1} = 1$$

$$\frac{5}{1} = 5$$

$$\frac{0}{1} = 0$$

$$\frac{-1}{1} = -1$$

$$\frac{1}{1} = 1$$

$$\frac{6}{1} = 6$$

y₃ value

$$OV - \frac{CKCE \times CKRE}{KE}$$

$$= \frac{1}{3} - \frac{\frac{1}{3} \times 1}{1} = 0$$

$$= \frac{2}{3} - \frac{\frac{1}{3} \times 5}{1} = -1$$

$$= 1 - \frac{\frac{1}{3} \times 0}{1} = 1$$

$$\frac{1}{6} - \frac{\frac{1}{3} \times -7}{1} = \frac{1}{2}$$

$$0 - \frac{\frac{1}{3} \times 1}{1} = -\frac{1}{3}$$

$$4 - \frac{\frac{1}{3} \times 6}{1} = 2$$

determination of the solution of the primal problem

Basic variable in the initial table of dual problem	S_1	S_2
$-(C_j - Z_j)$ of the final table of dual problem	0	$\frac{10}{3}$
corresponding primal duality	x_1	x_2

2) Solve the ~~LPP~~ following ~~LPP~~ & consider the following LPP & solve it using its dual solution

$$\min Z = 40x_1 + 30x_2 + 25x_3$$

STC

$$4x_1 + 2x_2 + 5x_3 \geq 30$$

$$3x_1 + 6x_2 + x_3 \geq 20$$

$$x_1 + 3x_2 + 6x_3 \geq 36$$

NNR

$$x_1, x_2, x_3 \geq 0$$

Sol Given cononical form
dual form

Let y_1, y_2, y_3 are the dual variables
 $\max Z = 30y_1 + 20y_2 + 36y_3$ [OF]

$$\begin{bmatrix} 4 & 2 & 5 \\ 3 & 6 & 1 \\ 1 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 3 & 1 \\ 2 & 6 & 3 \\ 5 & 1 & 6 \end{bmatrix}$$

STC

$$4y_1 + 3y_2 + y_3 \leq 40$$

$$2y_1 + 6y_2 + 3y_3 \leq 30$$

$$5y_1 + y_2 + 6y_3 \leq 25$$

NNR

$$x_1, x_2, x_3 \geq 0$$

cononical form duality

$$\max Z = 30y_1 + 20y_2 + 36y_3 + 0s_1 + 0s_2$$
 [OF]

STC

$$4y_1 + 3y_2 + y_3 + s_1 = 40$$

$$2y_1 + 6y_2 + 3y_3 + s_2 = 30$$

$$5y_1 + y_2 + 6y_3 + s_3 = 25$$

NNR

$$y_1, y_2, y_3 \geq 0$$

Simple initial tabu

CB _i	C _j	BD	20	36	0	0	0	Row	ratio
	BV	y ₁	y ₂	y ₃	s ₁	s ₂	s ₃		
0	s ₁	4	3	1	1	0	0	40	40
0	s ₂	2	6	3	0	1	0	30	10
0	s ₃	5	1	6	0	0	1	25	$\frac{25}{6} = 4.16$
	Z _j	0	0	0	0	0	0		
	Y-Z _j	30	20	36	0	0	0		

$\therefore S_3$ is leaving variable, y_3 is entering variable.

iteration-I

CB _i	C _j	30	20	36	0	0	0		
	BV	y_1	y_2	y_3	S_1	S_2	S_3	RDI	ratio
0	S_1	19/6	17/6	0	1	0	$-\frac{1}{6}$	$\frac{215}{6}$	$\frac{215}{7} = 12.6$
0	S_2	$-\frac{1}{2}$	$\frac{11}{2}$	0	0	1	$-\frac{1}{2}$	$\frac{35}{2}$	$\frac{35}{11} = 3.18$
36	y_3	$\frac{5}{6}$	$\frac{1}{6}$	1	0	0	$\frac{1}{6}$	$\frac{25}{6}$	25
25	S_3	30	6	36	0	0	6	150	
5-25		0	14	0	0	0	-6		

y_3 Values

$$\frac{5}{6} = \frac{5}{6}$$

$$\frac{1}{6} = \frac{1}{6}$$

$$\frac{6}{6} = 1$$

$$\frac{0}{6} = 0$$

$$\frac{0}{6} = 0$$

$$\frac{1}{6} = \frac{1}{6}$$

$$\frac{25}{6} = \frac{25}{6}$$

S_1 Values

$$DU - CKCE \times CKRE$$

$$= 4 - \frac{1 \times 5}{6} = \frac{19}{6}$$

$$= 3 - \frac{1 \times 1}{6} = \frac{17}{6}$$

$$= 1 - \frac{1 \times 6}{6} = 0$$

$$= 1 - \frac{1 \times 0}{6} = 1$$

$$= 0 - \frac{1 \times 0}{6} = 0$$

$$= 0 - \frac{1 \times 1}{6} = -\frac{1}{6}$$

$$= 40 - \frac{1 \times 25}{6} = \frac{215}{6}$$

S_2 values

$$= 2 - \frac{3 \times 5}{6} = -\frac{1}{2}$$

$$= 6 - \frac{3 \times 1}{6} = \frac{11}{2}$$

$$= 3 - \frac{3 \times 6}{6} = 0$$

$$= 0 - \frac{3 \times 0}{6} = 0$$

$$= 1 - \frac{3 \times 0}{6} = 1$$

$$= 0 - \frac{3 \times 1}{6} = -\frac{1}{2}$$

$$= 30 - \frac{3 \times 25}{6} = \frac{35}{2}$$

$\therefore$ optimality condition is not equal. All $C_j - Z_j$ values are not equal to '0' (or) '0'. Then we go to the iteration-II

$\therefore S_2$ is leaving variable

y_2 is entering variable

iteration - II

CB _i	C _j	30	20	36	0	0	0	sol	ratio
	BV	y ₁	y ₂	y ₃	s ₁	s ₂	s ₃		
0	y ₁	$\frac{113}{33}$	0	0	1	$-\frac{17}{33}$	$\frac{1}{11}$	$\frac{295}{11}$	$\frac{885}{113} = 7.83$
20	y ₂	$-\frac{1}{11}$	1	0	0	$\frac{2}{11}$	$-\frac{1}{11}$	$\frac{35}{11} = 3.18$	-35
36	y ₃	$\frac{28}{33}$	0	1	0	$-\frac{1}{33}$	$\frac{2}{11}$	$\frac{40}{11}$	$\frac{30}{7} = 4.28$
	Z _j	$\frac{316}{11}$	20	36	0	$\frac{28}{11}$	$\frac{52}{11}$	$\frac{2140}{11}$	
	C _j - Z _j	$\frac{14}{11} \uparrow$	0	0	0	$-\frac{28}{11}$	$-\frac{52}{11}$		

y₂ values

$$\frac{\frac{1}{2}}{\frac{11}{2}} = -\frac{1}{11}$$

$$\frac{\frac{11}{2}}{\frac{11}{2}} = 1$$

$$\frac{0}{\frac{11}{2}} = 0$$

$$\frac{0}{\frac{11}{2}} = 0$$

$$\frac{1}{\frac{11}{2}} = \frac{2}{11}$$

$$-\frac{\frac{1}{2}}{\frac{11}{2}} = -\frac{1}{11}$$

$$\frac{\frac{35}{2}}{\frac{11}{2}} = \frac{35}{11}$$

y₃ values

OV - CKRE x CKRE
RE

$$= \frac{5}{6} - \frac{\left(\frac{1}{6} \times -\frac{1}{2}\right)}{\left(\frac{11}{2}\right)} = \frac{28}{33}$$

$$= \left(\frac{1}{6}\right) - \frac{\left(\frac{1}{6} \times \frac{11}{2}\right)}{\left(\frac{11}{2}\right)} = 0$$

$$= 1 - \frac{\left(\frac{1}{6} \times 0\right)}{\left(\frac{11}{2}\right)} = 1$$

$$= 0 - \frac{\left(\frac{1}{6} \times 0\right)}{\left(\frac{11}{2}\right)} = 0$$

$$= 0 - \frac{\left(\frac{1}{6} \times 1\right)}{\frac{11}{2}} = -\frac{1}{33}$$

$$= \frac{1}{6} - \frac{\frac{1}{6} \times -\frac{1}{2}}{\frac{11}{2}} = \frac{2}{11}$$

$$= \frac{25}{6} - \frac{\frac{1}{6} \times \frac{35}{2}}{\frac{11}{2}} = \frac{40}{11}$$

S values

$$= 0 - \frac{CKCE \times CKRE}{KE}$$

$$= \frac{19}{6} - \frac{\frac{17}{6} \times -\frac{1}{2}}{\frac{11}{2}} = \frac{113}{33}$$

$$= \frac{17}{6} - \frac{\frac{17}{6} \times \frac{11}{2}}{\frac{11}{2}} = 0$$

$$= 0 - \frac{\frac{17}{6} \times 0}{\frac{11}{2}} = 0$$

$$= 1 - \frac{\frac{17}{6} \times 0}{\frac{11}{2}} = 1$$

$$= 0 - \frac{\frac{17}{6} \times 1}{\frac{11}{2}} = -\frac{17}{33}$$

$$= -\frac{1}{6} - \frac{\frac{17}{6} \times -\frac{1}{2}}{\frac{11}{2}} = \frac{1}{11}$$

$$= \frac{215}{6} - \frac{\frac{17}{6} \times \frac{35}{2}}{\frac{11}{2}} = \frac{295}{11}$$

∴ optimality condition all $g-z$ values are

not equal to 'zero' (or) '-ve'. so Then go to iteration III table.

y_3 is leaving variable

b_1 is entering variable

Iteration-III

CB _j	C _j	30	20	36	0	0	0		
	BV	y ₁	y ₂	y ₃	s ₁	s ₂	s ₃	101	ratio
0	s ₁	0	0	$-\frac{113}{28}$	1	$-\frac{11}{28}$	$-\frac{9}{14}$	$\frac{85}{7}$	
20	y ₂	0	1	$\frac{3}{28}$	0	$\frac{5}{28}$	$-\frac{1}{14}$	$\frac{25}{7}$	
30	y ₁	1	0	$\frac{33}{28}$	0	$-\frac{1}{28}$	$\frac{3}{14}$	$\frac{30}{7}$	
	z _j	30	20	$\frac{35}{2}$	0	$\frac{5}{2}$	5	200	
	cs-24	0	0	$-\frac{3}{2}$	0	$-\frac{5}{2}$	-5		

y₁ values

$$\frac{\frac{28}{33}}{\frac{28}{33}} = 1$$

$$\frac{0}{\frac{28}{33}} = 0$$

$$\frac{1}{\frac{28}{33}} = \frac{33}{28}$$

$$\frac{0}{\frac{28}{33}} = 0$$

$$\frac{-\frac{1}{33}}{\frac{28}{33}} = -\frac{1}{28}$$

$$\frac{\frac{2}{11}}{\frac{28}{33}} = \frac{3}{14}$$

$$\frac{\frac{40}{11}}{\frac{28}{33}} = \frac{30}{7}$$

y₂ values

$$01 - CKCE \times CKRE$$

$$= -\frac{1}{11} - \frac{-\frac{1}{11} \times \frac{28}{33}}{\frac{28}{33}} = 0$$

$$= 1 - \frac{-\frac{1}{11} \times 0}{\frac{28}{33}} = 1$$

$$= 0 - \frac{-\frac{1}{11} \times 1}{\frac{28}{33}} = \frac{3}{28}$$

$$= 0 - \frac{-\frac{1}{11} \times 0}{\frac{28}{33}} = 0$$

$$= \frac{2}{11} - \frac{-\frac{1}{11} \times -\frac{1}{33}}{\frac{28}{33}} = \frac{5}{28}$$

$$= -\frac{1}{11} - \frac{-\frac{1}{11} \times \frac{2}{11}}{\frac{28}{33}} = -\frac{1}{4}$$

$$= \frac{35}{11} - \frac{-\frac{1}{11} \times \frac{40}{11}}{\frac{28}{33}} = \frac{25}{7}$$

S_i values

DU - CKCE x CKRE

$$= \frac{113}{33} - \frac{113}{33} \times \frac{28}{33} = 0$$

$$= 0 - \frac{113}{33} \times 0 = 0$$

$$= 0 - \frac{113}{33} \times 1 = -\frac{113}{33}$$

$$= 1 - \frac{113}{33} \times 0 = 1$$

$$= -\frac{17}{33} - \frac{113}{33} \times -\frac{1}{33} = -\frac{11}{28}$$

$$= \frac{1}{11} - \frac{113}{33} \times \frac{2}{11} = -\frac{9}{14}$$

$$= \frac{295}{11} - \frac{113}{33} \times \frac{40}{11} = \frac{85}{7}$$

The optimality condition is true, All (G-2)' values are equal to the 'zero' (or) '-ve'.

UNIT-II

TRANSPORTATION PROGRAMMING PROBLEM

INTRODUCTION

The transportation problem deal with the distribution of a product manufactured at 'm' number of plants (or) factories (supply origins) to 'n' number of warehouses (or) retailers (demand destinations).

There are various types of transportation models and the simplest of them was first presented by F L Hitchcock in the year 1941. It was further developed by T C Koomans in the year 1949 and G B Dantzing in the year 1951. Several extensions of transportation model and methods have been subsequently developed.

Meaning

- "A shipment of commodities takes place from source to destination directly is called *transportation*."
- "The Transportation problem is to transport various amounts of a single homogeneous commodity that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum".

Example:

Three factories F1, F2 and F3 are producing the same product. From these factories, the product is distributed to three warehouses W1, W2 and W3. Each factory has a limited supply and each warehouse has specific demand. Each factory can transport to each warehouse but the transportation cost vary from different combination. The problem is to determine the quantity each factory should transport to each warehouse in order to minimum the total transportation cost.

MATHEMATICAL MODEL OF TPP

Let there be 'm' number of sources S1, S2,.....,Sm having a_i ($i=1,2,.....,m$) units of supply (or capacity) respectively, to be transported among 'n' number of destinations D1, D2,.....,Dn with b_j ($j=1,2,.....,n$) units of demand (or requirement) respectively.

Let ' c_{ij} ' be the cost of shipping one unit of commodity from source 'i' to destination 'j' for each route. If ' x_{ij} ' represents number of units shipped per route from source 'i' to destination 'j', the problem is to determine the transportation schedule so as to minimize the total transportation cost satisfying supply and demand conditions. Mathematically the transportation problem may be stated as follows:

Objective function: Minimize $Z = \sum_{i=1}^m \sum_{j=1}^n C_{ij} X_{ij}$

STC:

Supply constraint $\sum_{j=1}^n x_{ij} = a_i ; i=1,2,.....,m$

Demand constraint $\sum_{i=1}^m x_{ij} = b_j ; j=1,2,.....,n$

Restrictions: $x_{ij} > 0$ (non-negative variables)

Matrix form of TPP

		Destinations	Supply
		$D_1 D_2 D_3 \dots D_n$	
Sources	S_1	$c_{11} c_{12} c_{13} \dots c_{1n}$	a_1
	S_2	$c_{21} c_{22} c_{23} \dots c_{2n}$	a_2
	.	.	.
	.	.	.
	S_m	$c_{m1} c_{m2} c_{m3} \dots c_{mn}$	a_m
Demand		$b_1 b_2 b_3 \dots b_n$	$\sum a_i = \sum b_j$

SOLUTION OF TPP

The transportation problem can be solved in two stages as follows:

1. Initial Basic Feasible Solution (IBFS)
2. Optimal Basic Feasible Solution (OBFS)(or)Test for optimality

Note of Book:

- The following algorithm is used to obtain the solution of TPP.

TRANSPORTATION ALGORITHM**Step: 1: Formulation of TPP**

Formulate the problem and arrange the data in the matrix form ($m \times n$), where m = number of rows and n = number of columns.

Step: 2: Obtain an Initial basic Feasible Solution (IBFS)

The IBFS solution is obtained by any one the following method.

- a. North West Corner Method (NWCM)
- b. Least Cost Method (LCM)
- c. Vogel's Approximation Method (VAM) or Penalty Method

Step: 3: Test for optimality

Test whether the IBFS is optimal or not by using Modified Distribution (MODI) method.

Step: 4: Repeat the procedure

Go to step:3 and repeat the procedure until the optimal solution is reached.

Note of Book:

- Check the supply and demand and both should be same i.e. $\sum S_i = \sum D_j$, so that the problem is said to be as *balanced transportation problem*.
- If $\sum S_i \neq \sum D_j$, the problem is said to *unbalanced transportation problem*.

- The number of positive allocations must be equals to $(m+n-1)$, where 'm' is number of rows and 'n' is number of columns
 - The problem with $(m+n-1)$ allocations is said to *non-degeneracy transportation problem*.
 - The problem with less than $(m+n-1)$ allocations is said to *degeneracy transportation problem*.
- Cells in the transportation table having positive allocation will be called occupied cells, otherwise empty or non-occupied cells.

Initial Basic Feasible Solution (IBFS)

The IBFS solution is obtained by any one the following method.

- a. North West Corner Method (NWCM)
- b. Least Cost Method (LCM)
- c. Vogel's Approximation Method (VAM) or Penalty Method

North West Corner Method (NWCM) Algorithm

Step: 1: Select North-West corner cell of the given transportation table and allocate as many units as possible available supply and demand requirement.

Step: 2: If for any cell, supply equals to demand, then the next allocation can be made in cell either in the next row or column.

Step: 3: Continue the procedure until the total available quantity is fully allocated to the cells as required.

Step: 4: Determine the total transportation cost using $\sum C_{ij}X_{ij}$.

Least Cost Method (LCM)

Step: 1: Select the lowest transportation cost cell (minimum positive) among all the rows or columns of the transportation table and allocate as many units as possible available supply and demand requirement.

Step: 2: If the cells are tie, select the least cost cell, which has maximum possible from supply or demand

Step: 3: Repeat the procedure until the optimal solution is reached.

Step: 4: Determine the total transportation cost using $\sum C_{ij}X_{ij}$.

Vogel's Approximation Method (VAM) or Penalty Method

Step: 1: Calculate penalties for each row and column by taking the difference between least cost cell and next least cost cell of the given transportation table.

Step: 2: Identify the row or column which has maximum penalty and select least cost cell of the row or column and allocate as many units as possible available supply and demand requirement.

Step: 3: If there is a tie occurs in the value of penalty, select the least cost cell of that row or column. If there is a tie in the least cost cell also, then select the cost cell of row or column which with have maximum possible assignment.

Note of Book:

- If there is any row or column with zero supply or demand, then it should not use in calculating future penalties.

Step: 4: Repeat the above procedure until the optimal solution is reached.

Step: 5: Determine the total transportation cost using $\sum C_{ij}X_{ij}$.

Note of Book:

- The total cost relationship among NWCM, LCM and VAM is written as follows:
$$\text{NWCM} \geq \text{LCM} \geq \text{VAM}$$

Optimal Basic Feasible Solution (OBFS)

OBFS of transportation problem can be determined by using MODI method.

Modified Distribution Method

This method is best method and time saving over the stepping stone method for solving transportation problems. The following algorithm is used to test of optimality.

MODI ALGORITHM

Step: 1: Determination of IBFS

Determine the IBFS by using VAM (Penalty method)

Step: 2: calculation of U_i and V_j values

Calculate U_i and V_j values for each row and column by using occupied cells only. i.e. $C_{ij}=U_i+V_j$

Note of Book:

- Consider $U_i=0$ or $V_j=0$, where the maximum allocated cells at a row or column.
- If the equal number of allocations are available, select row or column which has maximum number of supply or demand.

Step: 3: Obtain opportunity cost

Determine the opportunity cost for unoccupied cells as follows:

$$d_{ij}=C_{ij} - U_i - V_j$$

Step: 4: Test for optimality

If the opportunity cost (d_{ij}) is greater than or equals to zero, then the solution is said to be an optimal solution i.e. $d_{ij} \geq 0$.

Note of Book:

- If one or more unoccupied cells have negative opportunity cost i.e. $d_{ij} < 0$, the solution is not optimal and can be improved.

Step: 5: Develop new solution

If the negative values are more than one cell, select the minimum negative value of unoccupied cell and make a star (*).

Step: 6: Draw a closed path or loop

Draw a closed loop or path from unoccupied (selected*) cell in clockwise direction and permitted only for occupied cells and ends with original unoccupied (selected*) cell.

Note of Book:

- Every closed path or loop must have even number of turns and it formed with horizontal and vertical lines only.

Step: 7: Develop new opportunity table

- a. Assign alternative signs positive (+) and negative (-) signs at the corner points of the closed path starting with a positive sign to the selected (*) cell.
- b. Select the negative sign cell which contains minimum allocation of units and then add that value to all positive sign cells and subtract that value from the negative sign cells.
- c. Determine the total transportation cost using $\sum C_{ij}X_{ij}$.

Step: 8: Repeat the procedure

Go to step:2, repeat the procedure until the optimal solution is reached.

UNBALANCED TPP

“If the sum of supply is not equal to sum of demand in the given matrix, then the problem is called unbalanced transportation problem”

Note of Book:

- If the supply is less than demand, then add the dummy row with zero (0) unit cost values.
- If the demand is less than supply, then add the dummy column with zero (0) unit cost values.
- Apply VAM to obtain IBFS.

ASSIGNMENT PROGRAMMING PROBLEM

INTRODUCTION

Assignment problem is used, when a particular job or machine is to be assigned to a particular worker based on his training and proficiency level or a project manager has to be deputed to a particular project based on his qualification, experience and special exposure to that special type of project.

Meaning

“An Assignment Problem is to assign ‘n’ number of origins (jobs) to the equal ‘n’ number of destinations (persons) at a minimum cost (or maximum profit)”.

Thus, the problem is how the assignments should be made so as to optimize the given objective.

Applications

Some of the problems where the assignment technique may be useful are:

- a. Assignment of workers to machines.
- b. Assignment of salesmen to different sales areas.
- c. Assignment of clerks to various checkout counters.
- d. Assignment of classes to rooms.
- e. Assignment of vehicles to routes.
- f. Assignment of contracts to bidders.

MATHEMATICAL MODEL OF APP

Let there be ‘n’ number of sources of facilities or workers $W_1, W_2, \dots, W_n$ having a_i ($i=1, 2, \dots, n$) units of supply (or capacity) respectively, to be assigned among ‘n’ number of activities or jobs $J_1, J_2, \dots, J_n$ with b_j ($j=1, 2, \dots, n$) units of demand (or requirement) respectively.

Let ‘ c_{ij} ’ be the effectiveness (cost, profit, time etc.) of assigning each facility to each job to determine the assignment schedule so as to optimize (maximize profit or minimize cost or time) the given objective. Mathematically the assignment problem may be stated as follows:

Objective function: Maximize or Minimize $Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$

STC:

Supply constraint $\sum_{j=1}^n x_{ij} = a_i = 1; i=1, 2, \dots, n$

Demand constraint $\sum_{i=1}^m x_{ij} = b_j = 1; j=1, 2, \dots, m$

Restrictions: $x_{ij} = 1$ (non-negative variables)

Assignment problem as special case of transportation problem

The assignment problem is seen to be the special case of transportation problem when each origin is associated with one and only one destination. In such case, $m=n$ and the numerical evaluations of such association are called ‘effectiveness’ instead of ‘transportation’. Mathematically, all a_i and b_j are unity (1), and each x_{ij} is limited to ‘1’.

Matrix form of APP

		Activities (jobs)	Supply
		$J_1 J_2 J_3 \dots J_n$	
Facilities or Workers	W_1	$c_{11} c_{12} c_{13} \dots c_{1n}$	1
	W_2	$c_{21} c_{22} c_{23} \dots c_{2n}$	1
	.	.	.
	.	.	.
	W_n	$c_{n1} c_{n2} c_{n3} \dots c_{nn}$	1
Demand		1 1 1 1	$\sum a_i = \sum b_j = n$

SOLUTION OF APP

Assignment problem can be solved by using any one of the following method.

1. Hungarian Method
 - a. Minimization
 - b. Maximization
 - c. Unbalanced Assignment problem
 - d. Travelling salesman method

Hungarian Method

This method is used to solve assignment problem so as to obtain optimal assignment schedule.

HUNGARIAN ALGORITHM**Step: 1: Formulation of APP**

Formulate the problem and arrange the data in the matrix form ($n \times n$), where n = number of rows = number of columns.

Step: 2: Determination of the Opportunity table

- a. Row operations: Locate the smallest element in each row of the given table and subtract that element from each element of that row.
- b. Column operations: Locate the smallest element in each column of the modified (*row operations*) table and subtract that element from each element of that column.

Note of Book:

- Each row and column must have at least one zero (0) value.

Step: 3: Making an Assignment

- a. Identify rows successively from top to bottom until a row with exactly one zero element is found.
- b. Make an assignment to single zero by making '0' box ☐
- c. For each zero value that becomes assigned, eliminate all other zeros in the same rows and columns by crossing the zero [X].
- d. Apply the above steps for each column also.
- e. Continue this procedure until all zeros in the rows and columns are either assigned or crossed off.

Step: 4: Test for optimality

If the number of assigned cells equals to the number of rows or columns then it is said to be optimal solution.

Hence the total value can be calculated by adding the original values of assigned cells from the given original table.

Note of Book:

- If the number of assignments are less than to number of rows or columns, then the solution is not optimal and can be improved.

Step: 5: Revise the opportunity table

Draw a set of horizontal and vertical lines to cover all the zeros in the revised table as follows:

- a. For each row in which no assignment was made, make a tick mark [✓].
- b. Evaluate the marked rows, if any zero cell occurs in these rows, make a tick mark [✓] to the respective columns that contains zeros.
- c. Evaluate the marked columns, if any assigned zeros occurs in those columns, make a tick mark [✓] to the respective rows that contains assigned zeros.
- d. Now draw straight lines through each marked column and such unmarked row.

Note of Book:

- If the number of lines drawn is equal to the number of rows or columns, the current solution is optimal.

Step: 6: Develop the new revised opportunity table

- a. Select the smallest element from the cells those not covered by any line and assumed as 'k'.
- b. Subtract that element 'k' from every element in the table those not covered by any line.
- c. Add that element to every element in the table those covered by the two lines i.e., intersection of two lines.
- d. Elements in the table covered by only one line remains same (unchanged)

Step: 7: Repeat the procedure

Go to step: 3, repeat the procedure until the optimal solution is reached.

Note of Book:

- Check the number of rows and number of columns both should be same, so that the problem is said to be as *balanced assignment problem*.
- If the number of rows is not equal to the number of columns, the problem is said to *unbalanced assignment problem*.
- The number of positive assignments must be equals to 'n', where $n = \text{number of rows} = \text{number of columns}$
- Cells in the assignment table having positive assignments will be called assigned cells, otherwise empty or non-assigned cells.

Minimization

This method is applied to obtain minimum cost or time required to complete a job.

Maximization

This method is applied to obtain maximum profit after completion of a job. The following steps are used to solve the maximum assignment problem.

Step: 1: Conversion of problem

Convert the given maximization problem into minimization problem by subtracting all cells from the maximum value (cell).

Step: 2: Apply Hungarian method to the minimization problem

Note of Book:

- Calculate total value by adding the original values from the given table.

Unbalanced Assignment problem

“If the number of rows is not equal to the number of columns in the given matrix, then the problem is called unbalanced assignment problem”

Note of Book:

- If the number of rows is less than number of columns, then add a row with zero unit values.
- If the number of columns is less than number of rows, then add a column with zero unit values.
- Apply Hungarian method to obtain optimal value.

Travelling Salesman Method

In this method a travelling salesman travel starting from his home city [the city from where he started], visiting each city only once and returning to his home city so that the total distance or cost or time is minimum.

Note of Book:

- There is no assignment for the same cities like; A-A, B-B etc.,

Mathematical model of travelling salesman method

		To Cities	Supply
		A BC.....n	
Form City	A	- $c_{12}c_{13}....c_{1n}$	1
	B	$c_{21} - c_{23}....c_{2n}$	1
	.		.
	.		.
	.		.
	n	$c_{n1}c_{n2}c_{n3}.... -$	1
Demand		1 1 1.....1	$\sum a_i = \sum b_j = n$

Note of Book:

- To solve the TSM, apply Hungarian method.

MCA: 201 Computer Oriented Operations Research

- If the sequence [**Eg:** A-B-C-D-E-A] is not reached, consider the least positive (>0) value from the cells as zero (0) and make an assignment.
- Repeat the above step until the sequence is reached.

Unit-2

Transportation problem [cost] [it is minimum problem]

Mathematical model of transportation problem:

Types of Transportation problems:-

Finding initial basic solution optimal solution by
UV method assignment problem, formation of assignment
problem - Hungarian method of solution, brach & bound
Technique for Assignment problem.

Introduction:

Transportation problem is a special type of ^{LPP} in
which goods are transported from a set of source to set of
destination subject to the supply & demand of the sources
& destination successfully such that the total cost of
transportation is minimizes.

→ The transportation problem is deal with the disk of a
product manufactured at ' m ' no. of plants (supply destination
not answers) to ' n ' no. of ware houses (demand destination
there are various types of transportation problem under
simplest of them was first present by Hitchcock.

→ In the year 1949 & GB Dantzig in the year 1951

→ Several Extensions after transportation problem model
& method have been subsequently developed.

Meaning:

A shipment of commodities takes place from
source to destination directly, is called Transportation.

Mathematical model of Transportation program (Tpp):

Let there be 'm' no. of sources $S_1, S_2, S_3, \dots, S_m$ having a_i ($i=1, 2, 3, \dots, m$) units of supply respectively to be Transportation among 'n' no. of destination $P_1, P_2, P_3, \dots, P_n$ with b_j ($j=1, 2, \dots, n$) units of demand respectively, commodity from source 'i' to destination 'j' for each route if x_{ij} represents no. of units shipped from source 'i' to destination. The problem is to determine the transportation schedule so as to minimize the total corporation schedule & also minimize the total transportation cost satisfying supply & demand conditions. Mathematically, the transportation problem may be stated as follows:

Objective function: $\text{Min } z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$

subject to constraints: supply constraints: $\sum_{j=1}^n x_{ij} = a_i; i=1, 2, 3, \dots, m$

demand constraints: $\sum_{i=1}^m x_{ij} = b_j; j=1, 2, 3, \dots, n$

Restrictions:

$x_{ij} \geq 0$ (non-negative variables)

matrix form of Tpp:

destination

		D_1	D_2	D_3	D_4	Supply
	Source					
	S_1	c_{11}	c_{12}	c_{13}	c_{14}	a_1
	S_2	c_{21}	c_{22}	c_{23}	c_{24}	a_2
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	S_m	c_{m1}	c_{m2}	c_{m3}	c_{m4}	$\sum a_i = \sum b_j$
	demand	b_1	b_2	b_3	b_4	

$m = \text{no. of sources}$

$N = \text{no. of destination}$

$a_i = \text{supply at the source } i$

$b_j = \text{demand at the destination } j$

$c_{ij} = \text{cost of transportation per unit from source } i \text{ to destination } j$

Types of Transportation problem:

Balanced = Sum of supplies = sum of demands

un Balanced = sum of supplies $\neq$ sum of demands

Methods of Transportation problem:

Phase: I

If finding the initial basic ^{feasible} solution [IBFS]

^{Types}
1) North-west corner cell

2) least cost cell method (or) matrix minima method

3) Vogel's approximation method.

(or)
Penalty method.

Phase: II

Optimization.

1) u-v method

(or)

MODI method

1) North-west corner cell method problems:

consider the transportation problem 3 sources & 4

destinations. The cell entries represents the cost of transportation per unit? Obtain the IBFS by using North-west corner "method", Least cost cell method,

VAM [Vogel's approximation method].

		destination				
		A	B	C	D	supply
Source	1	3	1	7	4	300
	2	2	6	5	9	400
	3	8	3	3	2	500
		250	300	400	200	demand

Sol North-west corner cell method

$$\sum a_i = \sum b_j$$

$$1200 = 1200 \quad [250 + 350 + 400 + 200 = 1200]$$

$$[300 + 400 + 500 = 1200]$$

Sum of Supplies = sum of demand

So the Given Transportation problem is Balance Transportation problem.

		destination				
		A	B	C	D	supply
Source	1	250 3	1	7	4	300
	2	2	6	5	9	400
	3	8	3	3	2	500
		250	350	400	200	demand

300	1	7	4	500
6	5	9	400	
3	3	2	500	
350	400	200		

200 6	5	9	400 100
3	3	2	500
300 0	400	200	

100 5	9	1000
3	2	500
400 200 300		

300 3	2	500 200
300 0	200	

200 2	200
200 0	

The final allocations are:

200 3	50 1	7	4
2	300 6	100 5	9
8	3	300 3	200 2

The total transportation cost is

$$\begin{aligned}
 &= (3 \times 250) + (1 \times 50) + (6 \times 300) + \\
 &\quad (5 \times 100) + (3 \times 200) + \\
 &\quad (2 \times 200) \\
 &= 4,100
 \end{aligned}$$

Least cost cell method: [IBFS]

1) destination

	1	2	3	4	supply
Source 1	3	1	7	4	300
2	2	5	5	9	400
3	8	3	3	2	500
	250	350	400	200	1200

demand

Q1 Given

For the given problem is least cost cell method

	1	2	3	4	
1	3	300	7	4	300 ⁰
2	2	5	5	9	400
3	8	3	3	2	500
	250	350	400	200	

50

$$300 - 350 = -50$$

1	250	5	5	9	150 ¹⁵⁰
2	8	3	3	2	500
	250	50	400	200	

0

$$400 - 250 = 150$$

	5	5	9	150
	3	3	200	500
	50	400	200	300

0

$$500 - 200 = 300$$

5	5	150
3	3	300
50	400	100

$$400 - 300 = 100$$

5	5	150
50	100	0

$$150 - 100 = 50$$

5	5	50
50	0	0

$$50 - 50 = 0$$

The final allocations are:

1	3	1	7	4	300
2	2	5	5	9	400
3	8	3	3	2	500
	250	350	400	200	

∴ The total transportation problem is:

$$(1 \times 300) + (2 \times 250) + (5 \times 50) + (5 \times 150) + (3 \times 300) + (2 \times 200) =$$

$$= 300 + 500 + 250 + 750 + 900 + 400 = 3100$$

2) Least cost method

	destination				
	1	2	3	4	supply
1	3	1	7	4	300
2	2	6	5	9	400
3	8	3	3	2	500
	250	300	400	200	
	demand				

8) Given

For the given problem is least cost method.

	1	2	3	4	supply
1	3	1	7	4	300
2	2	6	5	9	400
3	8	3	3	2	500
	250	350	400	200	
	demand				

$$300 - 350 = -50$$

	1	2	3	4	supply
1	3	6	5	9	400
2	2	3	3	2	500
	250	50	400	200	
	demand				

$$400 - 250 = 150$$

	1	2	3	4	supply
1	6	5	9		150
2	3	3	2		500
	50	400	200		
	demand				

$$500 - 200 = 300$$

	1	2	
1	8	5	150
2	3	3	300
	50	400	250

6	5
---	---

	6	5	150
	3	3	300
	50	400	250

5	150
3	250
400	150

5	150
150	

The final allocations are

3	500	7	4			
250	2	150	5	9		
8	300	3	250	3	200	2

∴ The total transportation problem is:

$$(1 \times 300) + (2 \times 50) + (5 \times 150) + (3 \times 300) + (250 \times 3) + (2 \times 200)$$

$$300 + 500 + 750 + 900 + 750 + 400 = 2850$$

Vogel's approximation method:

i)

		destination				
		1	2	3	4	supply
source	1	3	1	7	4	300
	2	2	6	5	9	400
	3	8	3	3	2	500
		250	350	400	200	
		demand				

Sol Given

Vogel's approximation method

$$\sum a_i = \sum b_j$$

$$1200 = 1200$$

∴ Sum of Supplies = Sum of demands

∴ It is a balanced transportation problem

					(Su)	
					RP/RD	row Penalty (row) difference
1	3	1	7	4	300	(2)
2	2	6	5	9	400	(3) ←
3	8	3	3	2	500	(1)
	250	350	400	200		
	(1)	(2)	(2)	(2)	CP/CD	(column differences)

<u>300</u> 1	7	4
6	5	9
3	3	2

0
300
150
500
RP/RD
(3) ←
(1)
(1)

highest value we can take the Row Penalty or column Penalty)

350
50
400
200

(2) (2) (2) CP/CL

6	5	9
3	3	<u>200</u> 2

150
500
300
RP/RD
(1)
(1)

50
400
200

(3) (2) (7) ↑ CP/CL

6	5
<u>50</u> 3	3

150
300
250
RP
(1)
(1)

50
400

(3) ↑ (2)

5
<u>250</u> 3

150
250
400
150

<u>150</u> 5

150
150
150

∴ The final allocation are

3	<u>300</u> 1	7	4
<u>250</u> 2	6	<u>150</u> 5	9
8	<u>50</u> 3	<u>250</u> 3	<u>200</u> 2

$$\therefore (300 \times 1) + (250 \times 2) + (150 \times 5) + (50 \times 3) + (250 \times 3) + (200 \times 2) = 2,850$$

Q2) find the IBFS for transportation using North-west cell method, least cost cell method, Penalty method

2	7	4	5
3	3	1	8
5	1	7	7
1	6	2	14
7	9	18	34

sol Given

North-west corner cell method

$$\sum a_i = \sum b_j$$

$$34 = 34$$

Sum of supply = sum of demand

$\therefore$ so it is balanced problem.

2	7	4	50
3	3	1	8
5	1	7	7
1	6	2	14
7	9	18	

2	3	1	86
5	1	7	7
1	6	2	14
70	9	18	

6	3	1	80
	1	7	7
	6	2	14
	Σ_3	18	

3	1	7	74
	6	2	14
	Σ_0	18	

4	7	140
	2	14
	Σ_0	14

2	2	14
	Σ_0	140

The final allocation of north-west corner is:

5	2	7	4
2	3	6	3
	5	3	1
	1	6	14

$$(5 \times 2) + (2 \times 3) + (6 \times 3) + (3 \times 1) + (4 \times 7) + (14 \times 2) = 93$$

② Given problem is least cost cell method

$$\sum a_i = \sum b_j$$

Sum of ^{supply} demand = sum of demand

$\therefore$ it is balanced transportation problem

2	7	4	5
8	8	8	80
5	1	7	7
1	6	2	14
7	9	18	10

2	7	4	5
5	7	7	70
1	6	2	14
7	9	10	2

2	7	4	5
7	6	2	14
7	2	10	0

7	4	5
6	2	0
2	10	3

7	3	4
2	30	0

2	7	0
7	0	0

The final allocation of least cost cell method

2	2	7	3	4
3		3	8	1
5	7	1		7
7	1	6	7	2

$$\begin{aligned}
 &= (7 \times 2) + (3 \times 4) + (7 \times 1) + (8 \times 1) + (7 \times 1) + (7 \times 2) \\
 &= 14 + 12 + 7 + 8 + 7 + 14 \\
 &= 62
 \end{aligned}$$

3) Given problem it is Vogel's approximation cell method.

$$\sum a_i = \sum b_j$$

Sum of supply = sum of demand
it is balanced transportation problem.

2	7	4	5 (2)
3	3	1	8 (2)
5	7	7	40 (4) ←
1	6	2	14 (1)
7	9	18	CP
(1)	(2)	(1)	

2	7	4	5 (2)
3	<u>2</u>	1	8 (2)
1	6	2	14 (1)
7	20	18	CP
(1)	(3)↑	(1)	

5	4	50 (2) ←
3	1	6 (2)
1	2	14 (1)
7	2	18
(1)	(1)	CP

R.P

3	1	6 (2)
2	2	12 (1)

20 18
(2) (1)

CP

R.P

6	60
2	12

18
12

12	120
2	

12

The final allocation of Vogel's approximation method.

5	2	7	4
3	2	3	6
5	7	1	7
2	1	6	12

$$= (5 \times 2) + (2 \times 3) + (6 \times 1) + (7 \times 1) + (2 \times 1) + (12 \times 2)$$

$$= 10 + 6 + 6 + 7 + 2 + 24$$

$$= 55$$

0	0	1	1	0	0
0	0	1	1	0	0
0	0	1	1	0	0
0	0	1	1	0	0

1) Consider the transportation problem

10	2	16	14	10	300
6	18	12	13	16	500
8	4	14	12	10	825
14	22	20	8	18	375
demand 300 400 250 150 400					

→ If find the IBFs using each of the following methods & compare their total cost.

1) North-west corner cell method

2) Least cost cell method

3) Vogel's approximation method.

Sol Given

Sum of supply = sum of demand

$$2000 \neq 1550$$

$$\Rightarrow 2000 - 1550 = 450$$

$\therefore$ It is not equal.

It is unbalanced problem.

$\Rightarrow$ Now we convert the unbalanced problem.

$\Rightarrow$ The demand side is column can be implemented as zero's.

10	2	16	14	10	0	300
6	18	12	13	16	0	500
8	4	14	12	10	0	825
14	22	20	8	18	0	375
300 400 250 150 400 450						

∴ It is a balanced problem.

1) North-west corner cell method:

Given

$$\sum a_i = \sum b_j$$

⇒ sum of supply = sum of demand

∴ It is balanced problem

300 10	2	16	14	10	0	300 0
6	18	12	13	16	0	500
8	4	14	12	10	0	825
14	22	20	8	18	0	375
350 50	400	250	150	400	450	

500 6	18	12	13	16	0	500 450
8	4	14	12	10	0	825
14	22	20	8	18	0	375
50 0	400	250	150	400	450	

400 8	12	13	16	0	400 50
14	14	12	10	0	825
22	20	8	18	0	375
400 0	250	150	400	450	

50 12	18	16	0	500
14	12	10	0	825
20	8	18	0	375
250 200	150	400	450	

20 14	12	10	0	825 625
20	8	18	0	375
200 0	150	400	450	

150 12	10	0	625 475
8	18	0	375
150 0	400	450	

400 10	0	475 75
18	0	375
400 0	450	

750 750	375
0	375
450 375	

$\Rightarrow$

375 375
375

$\therefore$ The final allocation of north-west corner cell method.

$$= (300 \times 10) + (50 \times 6) + (400 \times 18) + (50 \times 12) + (200 \times 14) + (150 \times 12) + (400 \times 10) + (75 \times 0) + (375 \times 0) =$$

$$= 3000 + 300 + 7200 + 600 + 2800 + 1800 + 4000 + 0 + 0$$

$$= 19,700/-$$

300 10	2	16	14	10	0
50 6	400 18	20 12	13	16	0
8	4	20 14	150 2	400 10	0
14	22	20	8	18	0

2) least cost cell method:

Given problem

$$\sum a_i = \sum b_j$$

sum of supply = sum of demand

$\therefore$ It is balanced transportation problem.

10	2	16	14	10	300	3000
6	18	12	13	16	0	500
8	4	14	12	10	0	825
14	22	20	8	18	0	375
350	400	250	150	400	150	

6	18	12	13	16	0	500
8	4	14	12	10	0	825
14	22	20	8	18	0	375
350	400	250	150	400	150	

6	18	12	13	16	350	
8	4	14	12	10	825	425
14	22	20	8	18	375	
350	400	250	150	400		

6	12	13	16	350	
8	14	12	10	425	
14	20	8	18	375	
350	250	150	400		

14	12	10	425
20	8	18	375 225

250 150 400
0

14	10	425 25
20	18	225

250 400
0

14	250
20	225

250
225

20	225
----	-----

The final allocation of Least cost cell method

10	2	16	14	10	300
350	6	18	12	13	150
8	4	25	14	12	100
14	22	225	20	150	8

$$\begin{aligned}
 & (350 \times 6) + (300 \times 0) + (150 \times 0) + (4 \times 400) + (25 \times 14) + (400 \times 10) + \\
 & (225 \times 20) + (150 \times 8) \\
 & = 2,100 + 0 + 0 + 1,600 + 350 + 4,000 + 4,500 + 1,200 \\
 & = 13,750
 \end{aligned}$$

3) Vogel's approximation method:

Given problem

$$\sum a_i = \sum b_j$$

Sum of supply = sum of demand

∴ It is a balanced transportation problem.

10	2	16	14	10	0	300	(2)
6	18	12	13	16	0	500	(6)
8	4	14	12	10	0	825	(4)
14	22	20	8	18	375 0	375 0	(8) ←
350	400	250	150	400	450	75	C.P
(2)	(2)	(2)	(4)	(0)	(0)		

10	2	16	14	10	0	300	(2)
6	18	12	13	16	35 0	500 425	(6) ←
8	4	14	12	10	0	825	(4)
350	400	250	150	400	750		C.P
(2)	(2)	(2)	(1)	(0)	(0)		

10	300 2	16	14	10	300 0	300 0	(8) ←
6	18	12	13	16		425	(6)
8	4	14	12	10		825	(4)
350	400 100	250	150	400			C.P
(2)	(2)	(2)	(1)	(0)			

6	18	12	13	16	425	<u>R.P</u> (6)
8	4	14	12	10	725	(14)

350 100 250 150 400
(2) (14) (2) (1) (6) C.PD

6	12	13	16	75	<u>R.P</u> (6) ←
8	14	12	10	725	(2)

350 250 150 400
(2) (2) (1) (6) C.P

12	13	16	75	<u>R.P</u> (1)
14	12	10	325	725 (2)

250 150 400
(2) (1) (6) ↑ C.P

12	13	75	<u>R.P</u> (1)
14	12	325	(2) ←

250 150
(2) (1)

12	75	750
14	175	
	250	175

14	175	1750
	175	

12,250

The final allocation of Vogel's approximation method:

10	<u>300</u>	2	16	14	10	0
<u>350</u>	6	18	<u>75</u>	12	13	<u>75</u>
8	<u>100</u>	<u>175</u>	<u>150</u>	<u>400</u>	10	0
14	22	20	8	18	<u>375</u>	0

$$\begin{aligned}
 &= (300 \times 2) + (350 \times 6) + (75 \times 12) + (100 \times 4) + (175 \times 14) + (150 \times 12) + \\
 &\quad (400 \times 10) + (75 \times 0) + (375 \times 0) \\
 &= 600 + 2,100 + 900 + 400 + 2,450 + 1,800 + 4,000 + 0 + 0 \\
 &= 12,250
 \end{aligned}$$

2) consider the following transportation problem.

5	1	8	7	5	15
3	9	6	7	8	25
4	2	7	6	5	42
7	11	10	4	9	35
30	20	15	10	20	

Q1 Given

$$\sum a_i = \sum b_j$$

Sum of supply = sum of demand

$$117 \neq 95$$

$$(22) 117 - 95 = 22$$

$\therefore$ It is unbalanced transportation problem.

To convert into balanced transportation problem.

5	1	8	7	5	0	15
3	9	6	7	8	0	25
4	2	7	6	5	0	42
7	11	10	4	9	0	35
30	20	15	10	20	22	

Given problem is North-west corner cell method.

$$\sum a_i = \sum b_j$$

Sum of supply = sum of demand

$$117 = 117$$

it is balanced transportation problem.

15	3	✓	8	7	5	0	150
3	9	6	7	8	0		25
4	2	7	6	5	0		42
7	11	10	4	9	0		35
30	20	15	10	20	22		
15							

15 3	9	6	7	8	0	15 10
4	2	7	6	5	0	42
7	11	10	4	9	0	35
15	20	15	10	20	22	

10	9	6	7	8	0	100
2	7	6	5	0		42
11	10	4	9	0		35
20	15	10	20	22		
10						

10	7	6	5	0	42 32
11	10	4	9	0	35

10 15 10 20 22

15	7	6	5	0	32 17
10	4	9	0		35
15	10	20	22		
0					

10	6	5	0	177
	4	9	0	35
	10	20	22	

7	5	0	70
	9	0	35
	20	22	

13	9	0	35	22
	13	22		

22	0	22
	22	0

The final allocation of north-west corner method.

15 3	1	8	7	5	0
15 3	10 9	6	7	8	0
4	10 2	15 7	10 6	7 5	0
15 7	11	10	4	13 9	21 0

$$\begin{aligned}
 \therefore & (15 \times 5) + (15 \times 3) + (10 \times 9) + (10 \times 2) + (15 \times 7) + (10 \times 6) + (7 \times 5) + \\
 & (13 \times 9) + (22 \times 0) = \\
 & = 75 + 45 + 90 + 20 + 105 + 60 + 35 + 117 + 0 \\
 & = 547
 \end{aligned}$$

2) Given problem it is least cost cell method.

$$\sum a_i = \sum b_j$$

$\therefore$ It is balanced transportation problem.

5	1	8	7	5	150	150
3	9	6	7	8	0	25
4	2	7	6	5	0	42
7	11	10	4	9	0	35
30	20	15	10	20	22	7

3	9	6	7	8	10	2518
4	2	7	6	5	0	42
7	11	10	4	9	0	35
30	20	15	10	20	70	

3	9	6	7	8	18
4	20	7	6	5	4222
7	11	10	4	9	35
30	20	15	10	20	

18 3	6	7	8	180
4	7	6	5	22
7	10	4	9	35
30	15	10	20	

12	7	6	5	22 10
7	10	4	9	35

12 15 10 20
0

7	6	5	10
10	<u>10</u> 4	9	35 25

15 10 20
0

7	10	5	100
10	9	25	

15 20 10

10	10	25 15
15	10	0

10	150
15	0

The final allocation problem of LCM

5	1	8	7	5	<u>15</u>	0		
<u>18</u>	3	9	6	7	8	<u>7</u>	0	
<u>12</u>	4	<u>20</u>	2	7	6	<u>10</u>	5	0
7	11	<u>15</u>	10	<u>10</u>	4	<u>20</u>	9	0

472

$$\begin{aligned}
 & (18 \times 3) + (12 \times 4) + (20 \times 2) + (5 \times 0) + (10 \times 4) + (10 \times 9) + (10 \times 5) + (7 \times 0) + (15 \times 0) \\
 & = 54 + 48 + 40 + 0 + 40 + 90 + 50 + 0 + 150 \\
 & = 472
 \end{aligned}$$

3) Given Problem is Vogel's Approximation cell Method.

5	1	8	7	5	0	15 ^{R.P} (1)
3	9	6	7	8	0	25 (3)
4	2	7	6	5	0	42 (2)
7	11	10	4	9	22 0	13 ³⁵ (4) ←
30	20	15	10	20	20	C.P
(1)	(1)	(1)	(2)	(0)	(0)	

5	15	8	7	5	15 ^{R.P} (4) ←
3	9	6	7	8	25 (3)
4	2	7	6	5	42 (2)
7	11	10	4	9	13 (3)
30	20	15	10	20	C.P
(1)	(1)	(1)	(2)	(0)	

3	9	6	7	8	25 (3)
4	5	7	6	5	37 ⁴² (2)
7	11	10	4	9	13 (3)
30	50	15	10	20	C.P
(1)	(7)	(1)	(2)	(3)	

R-P

25 3	6	7	8	25 (3) ←
4	7	6	5	37 (1)
7	10	4	9	13 (3)

30 5 15 10 20 C-P

(1) (1) (2) (3)

R-P

4	7	6	20 5	17 37 (1)
7	10	4	9	13 (3)

5 15 10 20 0 C-P

(3) (3) (2) (4) ↑

R-P

4	7	6	17 (2)
7	10	10 4	13 (3) ←

5 15 10 0 C-P

(3) (3) (2)

R-P

5 1	7	17 (3) ←
4	10	3 (3)

5 0 15 C-P

(3) (3)

$$\begin{array}{r} 128 \overline{) 17} \quad 120 \\ \underline{10} \quad 3 \\ 183 \end{array}$$

$$\begin{array}{r} 3 \overline{) 10} \quad 30 \\ 30 \end{array}$$

The final Allocation of Vogel's Approximation method:

5	<u>5</u> 1	8	7	5	0
<u>25</u> 3	9	6	7	8	0
<u>5</u> 4	<u>5</u> 2	7	6	<u>20</u> 5	0
<u>12</u> 7	11	<u>3</u> 10	<u>10</u> 4	9	<u>22</u> 0

$$\begin{aligned}
 &= (15 \times 1) + (25 \times 3) + (5 \times 4) + (5 \times 2) + (20 \times 5) + (12 \times 7) + (3 \times 10) + (10 \times 4) \\
 &\quad + (2 \times 0) \\
 &= 15 + 75 + 20 + 10 + 100 + 84 + 30 + 40 + 0 \\
 &= 374
 \end{aligned}$$

→ maximization case in transportation problem

Here the objective is to maximize the total profit for which the profit Matrix is given first we have to convert the maximization problem into minimization by subtracting all the elements from the highest element in the transportation problem.

→ Maximization Transportation problem is can be convert into minimization prblm.

→ By subtracting each transportation cost from maximum transportation cost

→ For Example the maximization transportation prblm is 25, so subtracted each value from 25.

There are certain types of transportation prblm where the objective function is to maximize the profit instead of minimization of cost. There problem can be solved by converting maximizing into minimizing.

1) Surya has three factories X, Y, Z. It's supplies to 4 dealers. It spread all over the country the production of these factors are 200, 500, 300 per month respectively.

	A	B	C	D	Supply
7	12	18	6	25	200
4	8	7	10	18	500
2	14	3	11	20	300
					Demand
					180 320 100 400

→ determine a suitable allocation to maximize the total net return.

Sol Given transportation problem, so convert maximize transportation into minimize transportation problem by subtracting each transportation cost from maximization of transportation of cost.
 $\therefore$ subtracting all transportation cost is 25.

13	7	19	0	200
17	18	15	7	500
11	22	14	5	300
180	320	100	400	1,000

$$25 - 12 = 13$$

$$\sum a_i = \sum b_j$$

Sum of supply = sum of demand

$$1000 = 1000$$

$\therefore$ It is balanced transportation problem.

Given problem North-west corner cell method

180 13	7	19	0	200 20
17	18	15	7	500
11	22	14	5	300
180	320	100	400	

20 7	19	0	200
18	15	7	500
22	14	5	300
320	100	400	

300 18	15	7	500 200
22	14	5	300
300	100	400	

100 15	7	200 100
14	5	300
100	400	

100 7	1000
5	300
400	

300 5	300
	0
300	

final allocation:

180 13	20 7	19	0
17	300 18	100 15	100 7
11	22	14	300 5

$$\begin{aligned}
 &\Rightarrow (180 \times 13) + (20 \times 7) + (300 \times 18) + \\
 &\quad (100 \times 15) + (100 \times 7) + (300 \times 5) \\
 &= 2,340 + 140 + 5,400 + \\
 &\quad 1,500 + 700 + 1,500 \\
 &= 11,580
 \end{aligned}$$

2) Given problem is least cost cell method

$$\sum a_i = \sum b_j$$

13 $\sum$ Sum of supply = sum of demand

→ It is a balanced transportation problem.

13	7	19	100	2000
17	18	15	7	500
11	22	14	5	300
180	320	100	400	200

17	18	15	7	500
11	22	14	5	300
180	320	100	200	100

17	18	15	500
100	22	14	100
180	320	100	0

17	18	100	500
80	320	100	0

80	7	18	400
80	320	0	0

320	18	320	0
320	0	0	0

final allocation problem.

13	7	19	<u>200</u> 0	200
<u>80</u> 17	<u>320</u> 18	<u>100</u> 15	7	500
<u>100</u> 11	22	14	<u>200</u> 5	300
180	320	100	200	

$$\begin{aligned}
 &= (80 \times 17) + (320 \times 18) + (100 \times 15) + (200 \times 0) + (100 \times 11) + (200 \times 5) \\
 &= 1,360 + 5,760 + 1,500 + 0 + 1,100 + 1,000 \\
 &= 10,720
 \end{aligned}$$

3) Given Vogel's approximation method problem.

$$\sum a_i = \sum b_j$$

Sum of supply = sum of demand

$\therefore$ It is balanced transportation problem

13	<u>200</u> 7	19	$\emptyset$	200 ^{R.P} (7)
17	18	15	7	500 (8)
11	22	14	5	300 (6)
180	320 120	100	400	
(2)	(11) $\uparrow$	(1)	(5)	<u>C.P</u>

17	18	15	400
11	22	14	5
180	120	100	400
(6)	(4)	(1)	(2)

100
500
300
C.P

17	18	15	100
180	22	14	120
180	120	100	300
(6)	(4)	(1)	(2)

18	15	100
22	14	120
120	100	20
(4)	(1)	

18	22
100	20

The final allocation of Vogel's Approximation method

13	200	7	19	0	200
17	100	18	15	400	500
180	20	22	14	5	300
180	320	100	400		

$$\begin{aligned}
 & \therefore (200 \times 7) + (100 \times 18) + (400 \times 7) + \\
 & (180 \times 11) + (20 \times 22) + (100 \times 14) = \\
 & = 1,400 + 1,800 + 2,800 + \\
 & 1,980 + 440 + 1,400 \\
 & = 9,820
 \end{aligned}$$

Transportation problem

I find the initial basic feasible solution in the first phase.

II It involves optimization of the initial basic feasible solution which is obtained in the phase-I

Finding the initial basic solution:-

1) North-west corner cell Method

2) Least cost cell Method

3) VAM [Vogel's approximation Method]

optimizing the Basic feasible solution applying UV Method.

1) Find the initial basic feasible solution of the following transportation problem by North-west corner cell Method & then optimize the solution using UV method.

	1	2	3	4	③
1	3	1	7	4	250
2	2	6	5	9	350
3	8	3	3	2	400
	200	300	350	150	

④

Sol I finding IBFS [initial basic feasible region (or) solution] using North-west corner cell method

$$\sum a_i = \sum b_j$$

Sum of supply = Sum of demand
1000 = 1000

<u>200</u> 3	1	7	4	250 50
2	6	5	9	350
8	3	3	2	400
200	300	350	150	
0				

<u>50</u> 1	7	4	50 0
6	5	9	350
3	3	2	400
300	350	150	
250			

<u>250</u> 6	5	9	350 100
3	3	2	400
250	350	150	
0			

<u>100</u> 5	9	100 0
3	2	400
350	150	
250		

<u>250</u> 3	2	150 0
250	150	
0		

<u>150</u> 2	180 0
150	

The final allocations are:

200	3	50	1	7	4
2	250	6	100	5	9
8	3	250	3	150	2

$$\begin{aligned}
 &= (200 \times 3) + (50 \times 1) + (250 \times 6) + (100 \times 5) + (250 \times 3) + (150 \times 2) \\
 &= 600 + 50 + 1,500 + 500 + 750 + 300 \\
 &= 3,700
 \end{aligned}$$

II. Application of u/v Method to optimize the solution.

$u_1 = 3$ $u_2 = 1$ $u_3 = 0$ $u_4 = -1$

$u_1 = 0$	200	3	50	1	7	4
$u_2 = 5$	2	250	6	100	5	9
$u_3 = 3$	8	3	250	3	150	2

row col.

row col. $m+n-1 = \text{no. of allocations}$

$$3+4-1 = 6$$

$6 = 6$ $\therefore$ it is exact.

Basic cells / Allocated cells

$$u_i + v_j = c_{ij}$$

$$\Rightarrow 0+3=3$$

$$0+1=1$$

$$5+1=6$$

$$5+0=5$$

$$3+0=3$$

$$3+1=2$$

NON basic cells / NON Allocated cells:

$$P_{ij} = u_i + v_j - c_{ij}$$

$$P_{31} = 3+3-8 = -2$$

$$P_{13} = 0+0-7 = -7$$

$$P_{32} = 3+1-3 = 1$$

$$P_{14} = 0+(-1)-4 = -5$$

$$P_{21} = 5+3-2 = 6 \uparrow$$

$$P_{24} = 5+1-9 = -3$$

(-) values to least the value

$x = 200$

+ add
- sub

iteration-1

$$V_1 = 0, V_2 = 0, V_3 = 0, V_4 = -1$$

$$U_1 = 0, U_2 = 5, U_3 = 3$$

3	1	7	4
2	6	5	9
8	3	3	2

Arrows indicate the path for finding the minimum value in the row and column of the selected cell (2,3) with value 5.

$$\begin{aligned}
 x &= 200 \\
 0 + \text{odd} &= 100 \\
 -\text{Sub} \\
 \Rightarrow 200 - 200 &= 0 \\
 \Rightarrow 200 + 50 &= 250 \\
 \Rightarrow 250 - 200 &= 50 \\
 \Rightarrow 0 + 200 &= 200
 \end{aligned}$$

Basic cells / Allocated cells

$$U_i + V_j = C_{ij}$$

$$U = 0$$

$$\Rightarrow 0 + 1 = 1$$

$$\begin{aligned}
 \Rightarrow 0 + 5 &= 5 \\
 5 + 1 &= 6 \\
 5 + (-3) &= 2
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow 3 + 0 &= 3 \\
 3 + (-1) &= 2
 \end{aligned}$$

Non-Basic cells / Non Allocated cells

$$P_{ij} = U_i + V_j - C_{ij}$$

$$P_{11} = 0 + (-3) - 3 = -6$$

$$P_{13} = 0 + 0 - 7 = -7$$

$$P_{14} = 0 + (-1) - 4 = -5$$

$$P_{24} = 5 + (-1) - 9 = -5$$

$$P_{31} = 3 + (-3) - 8 = -8$$

$$P_{32} = 3 + 1 - 3 = 1$$

$$P_{33} = 3$$

iteration - II

	$v_1 = -2$	$v_2 = 1$	$v_3 = 1$	$v_4 = 0$
$u_1 = 0$	3	1	7	4
$u_2 = 4$	2	6	5	9
$u_3 = 2$	8	3	3	2

$$\therefore x_1 = 50$$

$$50 - 50 = 0$$

$$50 + 100 = 150$$

$$50 - 250 = 200$$

$$50 + 0 = 50$$

Basic cells / Allocated cells

$$u_i + v_j = c_{ij}$$

$$u = 0$$

$$1^{st} \text{ row} \Rightarrow 0 + 1 = 1$$

$$2^{nd} \text{ row} \Rightarrow 4 + (-2) = 2$$

$$4 + 1 = 5$$

$$3^{rd} \text{ row} \Rightarrow 2 + 1 = 3$$

$$2 + 1 = 3$$

$$2 + 2 = 0$$

Non Basic cells / Non Allocated cells

$$P_{ij} = u_i + v_j - c_{ij}$$

$$P_{11} = 0 + (-2) - 3 = -5$$

$$P_{13} = 0 + 1 - 7 = -6$$

$$P_{14} = 0 + 0 - 4 = -4$$

$$P_{22} = 4 + 1 - 6 = -1$$

$$P_{24} = 4 + 0 - 9 = -5$$

$$P_{31} = 2 + (-2) - 8$$

$R_8 \therefore$ All P_{ij} values ≤ 0 , so optimality is reached.

$$\therefore (1 \times 250) + (200 \times 2) + (150 \times 5) + (50 \times 3) + (200 \times 3) + (150 \times 2)$$

$$250 + 400 + 750 + 150 + 600 + 300$$

$$= 2,450$$

2) Find the IBFS of the transportation problem by Penalty method & optimize the ^{solution} U.V method.

18	16	16	19	17	Supply
19	20	21	18	22	100
16	17	19	16	17	125
					175
Demand					60 80 85 105 70

80) Given problem is penalty method
 $\sum a_i = \sum b_j$

Sum of supply = sum of demand

$$400 = 400$$

∴ It is balanced transportation problem.

18	16	85 16	19	17	15	R.P
19	20	21	18	22	100	(0)
16	17	19	16	17	125	(1)
					175	(0)

60 80 ~~85~~ 105 70

(2) (1) (3) ↑ (2) (0)

18	16	19	17	15	R.P
19	20	18	22	100	(1)
60 16	17	16	17	125	(0)

60 80 105 70

(2) ↑ (1) (2) (0)

R.P

16	19	17	15 (1)
20	105 18	22	125 ²⁰ (2) ←
17	16	17	115 (1)
80	105	70	<u>C.P</u>
(1)	(2)	(0)	

R.P R.P

16	17	15 (1)
20	22	200 (2) ←
17	17	115 (0)
80	60	70 <u>C.P</u>
(1)	(0)	

R.P

16	17	150 (1) ←
17	17	115 (0)
60	70	<u>C.P</u>
(1)	(0)	

45	17	17	115
45	70		

70	17	70	0
70	0		

The final allocation is :

18	<u>15</u> 16	<u>85</u> 16	<u>105</u> 18	17
19	<u>20</u> 20	21	<u>105</u> 18	22
<u>60</u> 16	<u>45</u> 17	19	16	<u>70</u> 17

$$= (85 \times 16) + (20 \times 20) + (105 \times 18) + (60 \times 16) + (45 \times 17) + (15 \times 16) + (70 \times 17)$$

$$= 1360 + 400 + 1890 + 960 + 765 + 240 + 1190$$

$$= 6805$$

II Application of uv Method to optimize the solution.

$u_1 = 15, u_2 = 16, u_3 = 16, u_4 = 14, u_5 = 16$

$u_1 = 15$	18	<u>15</u> 16	<u>85</u> 16	19	17
$u_4 = 14$	19	<u>20</u> 20	21	<u>105</u> 18	22
$u_2 = 16$	<u>60</u> 16	<u>45</u> 17	19	16	<u>70</u> 17

$m+n-1 = \text{No. of Allocations}$

$$3+5-1=7$$

$$8-1=7$$

$$\boxed{7=7} \therefore \text{it is equal}$$

Basic cells (or) Allocated cells

$$u_i + v_j = c_{ij}$$

$$\text{1st row} \Rightarrow 0+15=16$$

$$0+16=16$$

$$\text{2nd row} \Rightarrow 4+16=20$$

$$4+14=18$$

$$\text{3rd row} \Rightarrow 15+1=16$$

$$16+1=17$$

$$16+1=17$$

Non Basic cells / Non Allocated cells:

$$P_{ij} = U_i + V_j - C_{ij}$$

$$P_{11} = 0 + 15 - 18 = -3$$

$$P_{14} = 0 + 14 - 19 = -5$$

$$P_{15} = 0 + 16 - 17 = -1$$

$$P_{21} = 4 + 15 - 19 = 0$$

$$P_{23} = 4 + 16 - 21 = -1$$

$$P_{25} = 4 + 16 - 22 = -2$$

$$P_{33} = 1 + 16 - 19 = -2$$

$$P_{34} = 1 + 14 - 16 = -1$$

∴ So All $P_{ij} \leq 0$, so optimality is reached

$$\therefore (15 \times 16) + (85 \times 16) + (20 \times 20) + (105 \times 18) + (60 \times 16) + (45 \times 17) + (60 \times 17) \\ = 6805$$

3) Solve the transportation problem (IBFS & UV)

	1	2	3	Supply
A	2	7	4	5
B	3	3	1	8
C	5	4	7	7
D	1	6	2	14
Demand	7	9	18	

So Given

Transportation problem is balanced problem.

Sum of supply = Sum of demand

$$\sum a_i = \sum b_j$$

$$34 = 34$$

North-west corner cell method

<u>5</u> 7	7	4	50
3	3	1	8
5	4	7	7
1	6	2	14
7	9	18	

<u>2</u> 3	3	1	86
5	4	7	7
1	6	2	14
70	9	18	

<u>6</u> 3	1	60
4	7	7
6	2	14
9	18	

3 4	7	7 4
6	2	14

~~3~~0 18

<u>14</u> 7	40
2	14
18	14

<u>14</u> 2	40
140	

∴ The final allocation are:

<u>5</u> 2	7	4	10
<u>2</u> 3	<u>6</u> 3	1	19
5	<u>3</u> 4	<u>14</u> 7	51
1	6	<u>14</u> 2	81

$$\begin{aligned}
 &\therefore (5 \times 2) + (2 \times 3) + (6 \times 3) + (3 \times 4) + (14 \times 7) + (14 \times 2) \\
 &= 10 + 6 + 18 + 12 + 98 + 28 \\
 &= 172
 \end{aligned}$$

(ii) Application of u-v method to optimize the solution.

$V_1 = 2 \quad V_2 = 2 \quad V_3 = 5$

$U_1 = 0$	5	2	7	4
$U_2 = 1$	2	3	6	3
$U_3 = 2$		5	3	4
$U_4 = -3$	1		6	2

$x = 6$
 (+) add
 (-) sub

$m + n - 1 = \text{No. of allocations}$

$$4 + 3 - 1 = 6$$

$$7 - 1 = 6$$

$$6 = 6$$

∴ It is equal.

Basic cells / Allocated cells

$$U_i + V_j = C_{ij}$$

$$\Rightarrow 0 + 2 = 2$$

$$\Rightarrow 1 + 2 = 3$$

$$1 + 2 = 3$$

$$\Rightarrow 2 + 2 = 4$$

$$2 + 5 = 7$$

$$\Rightarrow -3 + 5 = 2$$

Non-Basic cells / Non Allocated cells

$$P_{ij} = U_i + V_j - C_{ij}$$

$$P_{12} = 0 + 2 - 7 = -5$$

$$P_{13} = 0 + 5 - 4 = 1$$

$$P_{23} = 1 + 5 - 1 = 5$$

$$P_{31} = 2 + 2 - 5 = -1$$

$$P_{41} = -3 + 2 - 1 = -2$$

$$P_{42} = -3 + 2 - 6 = -7$$

$\therefore$ So $P_{ij} \neq 0$, so optimality condition is not
 equal, then we go to iteration-I

$$v_1 = 2 \quad v_2 = -3 \quad v_3 = 0$$

$u_1 = 0$	5 2	7	4
$u_2 = 1$	2 3	3	6 1
$u_3 = 7$	5	9 4	8 7
$u_4 = 2$	1	6	14 2

Basic cells (Allocated cells)
 $u_i + v_j = c_{ij}$

NON-Basic cells (NON Allocated cells):

$$P_{ij} = u_i + v_j - c_{ij}$$

$$P_{12} = 0 + (-3) - 7 = -10$$

$$P_{13} = 0 + 0 - 4 = -4$$

$$P_{22} = 1 + (-3) - 3 = -5$$

$$P_{31} = 7 + 2 - 5 = 4 \uparrow$$

$$P_{41} = 2 + 2 - 1 = 3$$

$$P_{42} = 2 + (-3) - 6 = -7$$

$\therefore$ So $P_{ij} \neq 0$, so optimality condition is not
 equal, then we go to iteration-II

	$u_1 = 2$	$u_2 = 1$	$u_3 = 4$	
$u_1 = 0$	5 2	7	8 1	$P_{12} = -6$
$u_2 = -3$	3	6	6 7	$P_{13} = 0$
$u_3 = 3$	2 5	9 4	14 2	$P_{21} = -4$
$u_4 = -2$	1	6		$P_{22} = -8$
				$P_{41} = -1$
				$P_{42} = -7$

Non-Basic cells / Non Allocated cells

$$P_{ij} = u_i + v_j - c_{ij}$$

$$P_{12} = 0 + 1 - 7 = -6$$

$$P_{13} = 0 + 4 - 4 = 0$$

$$P_{21} = -3 + 2 - 3 = -4$$

$$P_{22} = -3 + 1 - 6 = -8$$

$$P_{31} = -2 + 2 - 1 = -1$$

$$P_{42} = -2 + 1 - 6 = -7$$

$\therefore$ So $P_{ij} = 0$, so optimality condition is true.

$\therefore$ Hence it is proved.

1	2	3
1	2	3
4	1	2
2	1	1

Assignment Transportation problems : (minimum problem)

Types of Assignment problem:

Balanced Assignment problem [No. of rows = No. of columns]

unBalanced Assignment problem [No. of rows $\neq$ No. of columns]

Hungarian Method:

Phase I :- Rows & column Reductions:

step 1: Subtract the minimum value of each row from the entries of that row.

step 2: Subtract the minimum value of each column from the entries of that column.

Phase II :- Optimization of the problem:

step 1: Draw a No. of minimum lines to cover all the '0' of the matrix.

(a) rows Scanning

(b) column Scanning

step 2: Check whether the no. of square marked is equal to the no. of rows of the matrix, if yes go to step 5, otherwise go to step 3.

step 3: Identify the minimum undeleted ^{cell} values. at the Identify the minimum value's of the undeleted cell values.

(a) Add the minimum undeleted values at the intersection points of the present matrix.

(b) Subtract the minimum undeleted values from all the undeleted cell values.

(c) All other entries remains same.

step 4: Go to step ①

step 5: Treat the solution as marked by the square is the optimal solution.

1) Solve the following Assignment problem using the Hungarian Method. the matrix entries represents the Processing time & hrs.

Job	operator				
	1	2	3	4	5
1	9	11	14	11	7
2	6	15	13	13	10
3	12	13	6	8	8
4	11	9	10	12	9
5	7	12	14	10	14

Sol: The Given

Assignment problem is Balanced.

No of rows = No of column

$$5 = 5$$

Phase I: Identification of minimum row number

	1	2	3	4	5	<u>R.M</u>
1	9	11	14	11	7	7 [Row least value]
2	6	15	13	13	10	6
3	12	13	6	8	8	6
4	11	9	10	12	9	9
5	7	12	14	10	14	7

Row Reduction matrix

2	4	7	4	0
0	9	7	7	4
6	7	0	2	2
2	0	1	3	0
0	5	7	3	7

[least values of column]

column Reduction matrix

2	4	7	2	0
0	9	7	5	4
6	7	0	0	2
2	0	1	1	0
0	5	7	1	7

$$\begin{aligned} 7-9 &= 2 \\ 7-11 &= 4 \\ 7-14 &= 7 \\ 7-11 &= 4 \\ 7-7 &= 0 \end{aligned}$$

$$\begin{aligned} \text{1st col.} \\ 0-2 &= 2 \\ 0-0 &= 0 \\ 0-6 &= 6 \\ 0-2 &= 0 \\ 0-0 &= 0 \\ \text{2nd col.} \\ 0-4 &= 4 \\ 0-9 &= 9 \end{aligned}$$

Row Scanning & Column Scanning

2	4	7	2	0
0	9	7	5	4
6	7	0	0	2
2	0	1	1	0
0	5	7	1	7

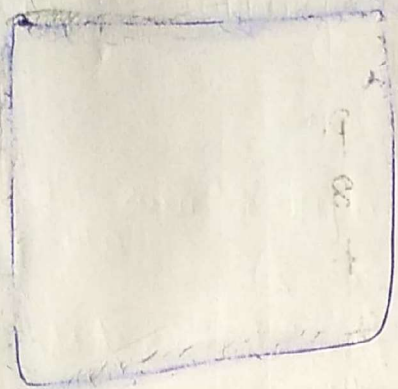
→ leaving least value = 1

$x=1$ ⊕ intersection (add)
 $x=1$ — undeleted (-)
 $x=1$ — deleted [copy]

iteration-I

2	4	6	1	0
0	9	6	4	4
7	8	0	0	3
2	0	0	0	0
0	5	6	0	7

∴ The No. of squares are
 Not equal to the job & operator
 so then go to iteration-II



∴ From iteration I table the
No. of jobs & operator can be
equal.

<u>Job</u>	<u>operator</u>	<u>time</u>	<u>time</u>
1 →	5 →	8	7
2 →	1 →	7	6
3 →	3 →	7	6
4 →	2 →	10	9
5 →	4 →	11	10
		<u>43 hrs</u>	<u>38 hrs</u>

2) Solve the following problem by using Hungarian method.

		operator				
		1	2	3	4	5
job	1	10	12	15	12	8
	2	7	16	14	14	11
	3	13	14	7	9	9
	4	12	10	11	13	10
	5	8	13	15	11	15

Sol Given

The Assignment problem is Balanced.

No. of rows = No. of columns

$$\boxed{5 = 5}$$

∴ So, it is balanced Assignment Problem.

Phase: I

Identification of minimum row Number

	1	2	3	4	5	R.M
1	10	12	15	12	8	8
2	7	16	14	14	11	7
3	13	14	7	9	9	7
4	12	10	11	13	10	10
5	8	13	15	11	15	8

Row Reduction method

2	4	7	4	0
0	9	7	7	4
6	7	0	2	2
2	0	1	3	0
0	5	7	3	7
0	0	0	2	0

(least values of col.)

Column Reduction Method

2	4	7	2	0
0	9	7	5	4
6	7	0	0	2
2	0	1	1	0
0	5	7	1	7

Row Scanning & column Scanning

2	4	7	2	0
0	9	7	5	4
6	7	0	0	2
2	0	1	1	0
0	5	7	1	7

$x=1$ $\left\{ \begin{array}{l} \text{intersection (+)} \\ \text{undetected (-)} \\ \text{deleted (copy)} \end{array} \right.$

Here, the Square boxes are not equal to the No. of jobs & operator, so then we go to the iteration(I).

Iteration-I

2	4	6	1	0
0	9	6	4	4
7	8	0	0	3
2	0	0	0	0
0	5	6	0	7

$\therefore$ The No. of job & No. of Operators of the Square boxes can be equal.

<u>Job</u>	<u>operator</u>	<u>time</u>
1 $\rightarrow$ 5 $\Rightarrow$		8
2 $\rightarrow$ 1 $\Rightarrow$		7
3 $\rightarrow$ 3 $\Rightarrow$		7
4 $\rightarrow$ 2 $\Rightarrow$		10
5 $\rightarrow$ 4 $\Rightarrow$		11
		<u>43 hrs</u>

unbalanced Assignment Problem

The no. of rows is not $\neq$ to the no. of columns then assignment problem is called unbalanced. so we convert unbalanced assignment by the ~~un~~ balanced assignment problem by adding dummy row & dummy column.

i) solve the following assignment problem by using Hungarian Method.

	Subject				
	1	2	3	4	5
1	30	39	31	38	40
2	43	37	32	35	38
3	34	41	33	41	34
4	39	36	43	32	36
5	32	49	35	40	37
6	36	42	35	40	42

Sol Given

The problem is assignment it is unbalanced.

The NO. of rows $\neq$ NO. of columns

The Given transportation problem is unbalanced trans-
portation prblm. so, we convert unbalanced into the
balanced transportation prblm by adding dummy rows
& dummy column.

	1	2	3	4	5	6	Row Min (R _m)
1	30	39	31	38	40	0	0
2	43	37	32	35	38	0	0
3	34	41	33	41	34	0	0
4	39	36	43	32	36	0	0
5	32	49	35	40	37	0	0
6	36	42	35	40	42	0	0

Row Reduction method:

30	39	31	38	40	0
43	37	32	35	38	0
34	41	33	41	34	0
39	36	43	32	36	0
32	49	35	40	37	0
36	42	35	40	42	0

column
least
value

30	36	31	³² 35	34	0
----	----	----	------------------	----	---

column reduction method & row scanning &

0	3	0	6	6	0
13	1	1	3	4	0
4	55	2	9	0	0
9	0	12	0	2	0
2	13	4	8	3	0
6	6	4	8	8	0

column scanning

$2=1$ is least value
 $x=1$ ← intersection (+)
 undelcted (-)
 delcted copy

∴ Here the No. of faculty & subjects are to be the not equal of squares, then go to iteration-I

iteration-I

0	3	0	6	7	1	x
12	0	0	2	4	0	x
3	4	1	8	0	0	x
9	0	12	0	3	1	x
1	12	3	7	3	0	x
5	5	3	7	8	0	x

$x=1$ (leaving least value)
 $x=1$ (f)
 $x=1$ (t)
 $x=1$ (copy)

iteration-II

0	3	0	6	7	2	x
12	0	0	2	4	1	x
3	4	1	8	0	1	x
9	0	12	0	3	2	x
0	11	2	6	2	0	x
4	4	2	6	7	0	x

∴ Here the faculty & subjects equal No. of squares it is equal.

faculty subject time

1 → 3 ⇒ 31

2 → 2 ⇒ 37

3 → 5 ⇒ 34

4 → 4 ⇒ 32

5 → 1 ⇒ 32

6 → 6 ⇒ 0

166 hrs

31	37	34	32	32	0
31	37	34	32	32	0
31	37	34	32	32	0
31	37	34	32	32	0
31	37	34	32	32	0
31	37	34	32	32	0

Assignment problems:

We convert maximization problem into the minimization problem by subtracting all the elements from the largest element.

1) Solve the following Assignment problem by using Hungarian method to maximum sales.

	Territories			
	1	2	3	4
A	45	38	30	22
B	35	29	20	14
C	35	29	20	14
D	27	20	15	10

sol Given

Assignment problem is maximization problem type. So, we convert maximization into minimization by subtract maximum value from all the elements.

	1	2	3	4	R.M
A	0	7	15	23	0
B	10	16	25	31	10
C	10	16	25	31	10
D	18	25	30	35	18

$$\max = 45$$

$$45 - 45 = 0$$

$$45 - 38 = 7$$

$$30 - 45 = 15$$

Row reduction method

0	7	15	23
0	6	15	21
0	6	15	21
0	7	12	17
0	6	12	17

col. least value

column Reduction method & Row Scanning & column scanning method

0	1	3	6
0	0	3	4
0	0	3	4
0	1	0	0

$x=3$ $x=3$ $\begin{matrix} (+) \\ (-) \\ (copy) \end{matrix}$

Iteration-I

0	1	0	3
0	0	0	1
0	0	0	1
3	1	0	0

$\therefore$ Here we can use the Application of diagonal selection

$\therefore$ Hence the No. of sales man & territories of the squares are equal [No. of rows = No. of columns]
Hence it is proved.

<u>sales man</u>	<u>Territories</u>	<u>time</u>
A $\Rightarrow$	1	0
B $\Rightarrow$	2	16
C $\Rightarrow$	3	25
D $\Rightarrow$	4	35
		<u>76 hrs</u>

Assignment problem - Branch & Bound Method:

Solve the following Assignment problem by using branch & bound Method.

	1	2	3	4
a	9	2	7	8
b	6	4	3	7
c	5	8	1	8
d	7	6	2	4

Sol Given

The Assignment problem is balanced Assignment problem.

No. of rows = No. of columns

	1	2	3	4
a	9	2	7	8
b	6	4	3	7
c	5	8	1	8
d	7	6	2	4

LB = Lower bound

$$LB = 2 + 3 + 1 + 2 = 8$$

$\Rightarrow$ [every row, we can take least value]

$LB = 2+3+1+2 = 8$

$a=1$

$9+4+1+4=18$

$a=2$

$2+5+1+4=12$

least value

$a=3$

$7+5+4+4=20$

$a=4$

$8+5+4+1=18$

$a=2$

$b=1$

$6+2+1+4=13$

$b=3$

$3+5+2+4=14$

$b=4$

$7+5+2+1=15$

$a=2, b=1, c=3, d=4$

$2+6+1+4=13$

$a=2, b=1, c=4, d=3$

$2+6+8+2=18$

Time

$a \rightarrow 2 \rightarrow 2$

$b \rightarrow 1 \rightarrow 6$

$c \rightarrow 3 \rightarrow 11$

$d \rightarrow 4 \rightarrow 4$

$UB: 13hrs$

(upper bound)

$LB = 8$

$UB = 13$

2) Solve the following Assignment problem by using branch & bound Method.

	1	2	3	4
1	23	20	21	24
2	19	21	20	20
3	20	18	24	22
4	22	18	21	23

Sol Given

The Assignment problem is balanced Assignment Problem.

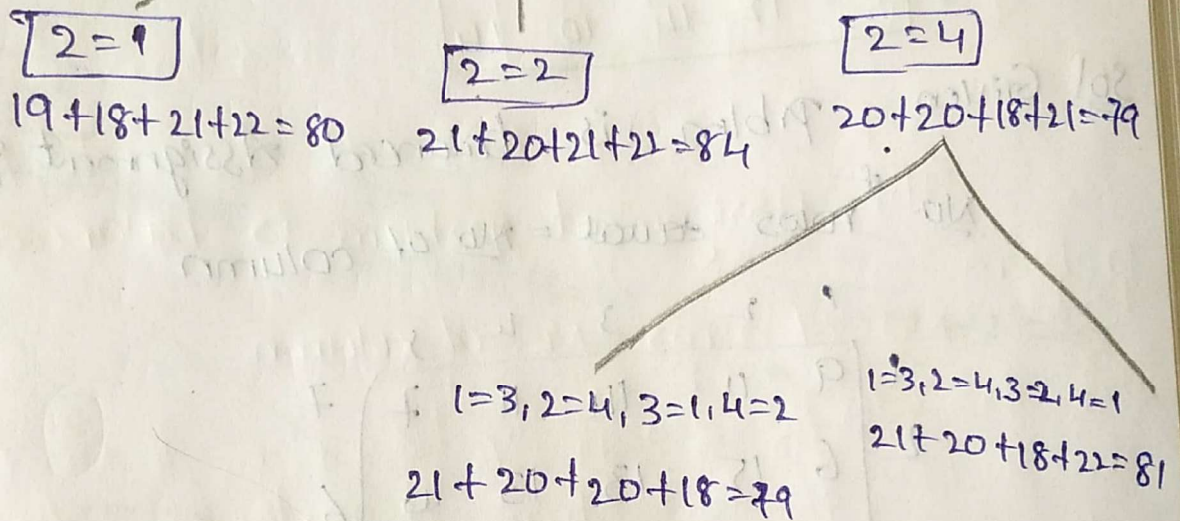
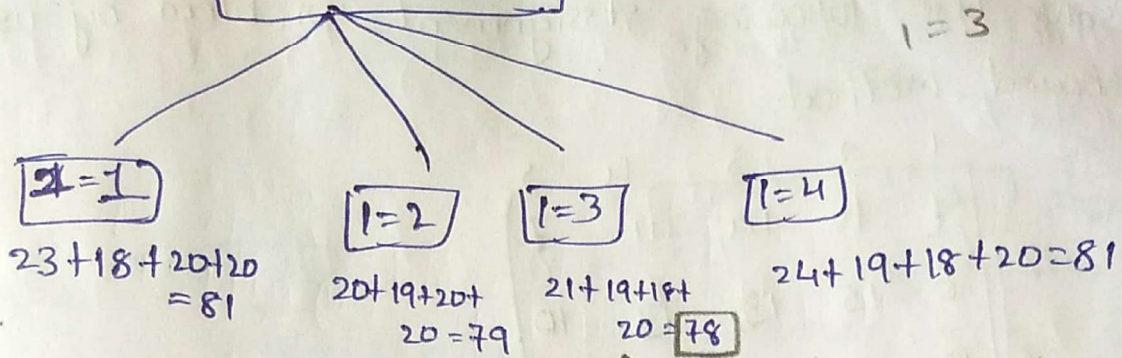
No. of rows = No. of columns.

	1	2	3	4
1	23	20	21	24
2	19	21	20	20
3	20	18	24	22
4	22	18	21	23

LB = Lower Bound

$$LB = 20 + 19 + 18 + 18 = 75$$

$$LB = 20 + 19 + 18 + 18 = 75$$



Job operator time

$$1 \rightarrow 3 = 21$$

$$2 \rightarrow 4 = 20$$

$$3 \rightarrow 1 = 20$$

$$4 \rightarrow 2 = 18$$

$$UB = 79 \text{ hrs}$$

$$LB - UB = 75 \text{ to } 79$$

3) solve the following Assignment problem by using brand & bound method.

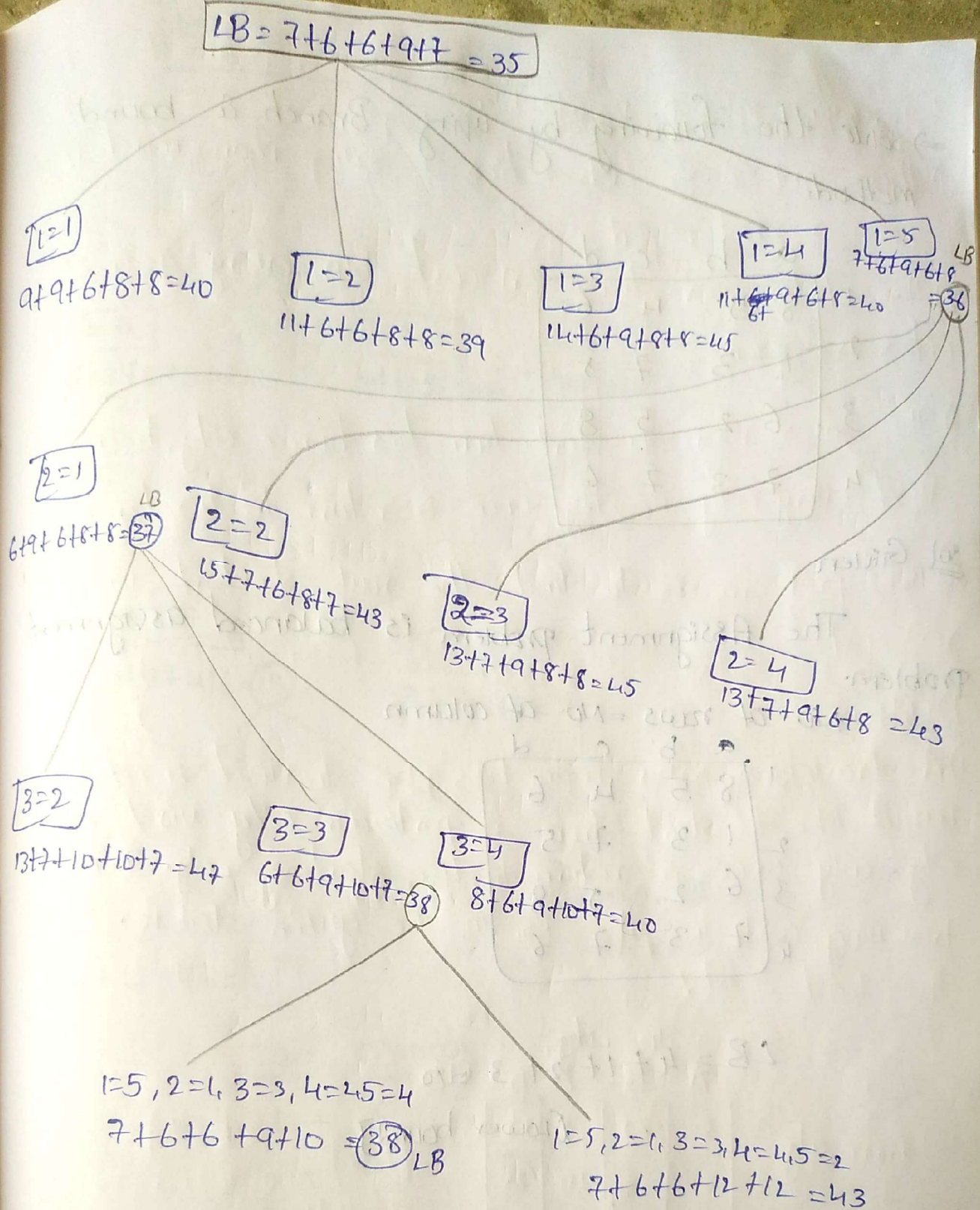
	1	2	3	4	5
1	9	11	14	11	7
2	6	15	13	13	10
3	12	13	6	8	8
4	11	9	10	12	9
5	7	12	14	10	14

Sol Given problem is balanced Assignment problem.
 No. of rows equal = No. of column

	1	2	3	4	5	
1	9	11	14	11	7	7
2	6	15	13	13	10	6
3	12	13	6	8	8	6
4	11	9	10	12	9	9
5	7	12	14	10	14	7

$$7 + 6 + 6 + 9 + 7 = 35$$

[Lower bound]



Job operator time

1 - 5 - 7

2 - 1 - 6

3 - 3 - 6

4 - 2 - 9

5 - 4 - 10

38 U.B

→ LB to UB
35 to 38

⇒ Solve the following by using Branch & bound Method.

	a	b	c	d
1	8	5	4	6
2	1	3	7	5
3	6	2	5	8
4	7	3	7	6

Sol Given

The Assignment problem is balanced assignment problem.

No. of rows = No. of column

	a	b	c	d
1	8	5	4	6
2	1	3	7	5
3	6	2	5	8
4	7	3	7	6

$$LB = 4 + 1 + 2 + 3 = 10$$

[Lower bound]

$$LB = 4 + 1 + 2 + 3 = 10$$

$$1 = a$$

$$8 + 2 + 5 + 5 = 20$$

$$1 = b$$

$$5 + 1 + 5 + 5 = 16$$

$$1 = c$$

$$4 + 1 + 2 + 5 = 12$$

$$1 = d$$

$$6 + 1 + 2 + 5 = 14$$

$$2 = a$$

$$1 + 2 + 4 + 6 = 13$$

$$2 = b$$

$$3 + 6 + 4 + 6 = 19$$

$$2 = d$$

$$5 + 6 + 2 + 4 = 17$$

$$1 = c, 2 = a, 3 = b, 4 = d$$

$$4 + 1 + 2 + 6 = 13$$

$$1 = c, 2 = a, 3 = d, 4 = b$$

$$4 + 1 + 8 + 3 = 16$$

Job operator time

1	c	4
2	a	1
3	b	2
4	d	6

13 upper bound

LB to UB

10 to 13

Travelling Sales Man Problem:

1) There are a no. of cities a sales man must visit

Ex: a b c d.

2) The distance (or) time (or) cost b/w every pair of cities is given.

3) The sales man ~~to~~ start from his home city, he must visit every city exactly once & return to his home city.

4) The problem is to find the route shortest (or) distance (or) cost.

Problem:

1) A travelling sales man has plan to visit 4 cities. He would like to start from a particular city visit each city only once & return to the starting city. The travelling cost in the Rs. in the given table. Find the least cost?

		To city				RM
		A	B	C	D	
from city	A	0	25	75	45	
	B	35	0	150	25	
	C	35	40	0	15	
	D	65	75	130	0	

Sol Given

problem is Assignment problem it is balanced.

	A	B	C	D	Rm
A	2	25	75	45	25
B	35	2	150	25	25
C	35	40	2	15	15
D	65	75	130	2	65

Row reduction method:

0	50	20
10	125	0
20	25	0
0	10	65

	A	B	C	D
A	-	0	50	20
B	10	-	125	0
C	20	25	-	0
D	0	10	130	-
	0	0	50	0

column Reduction method

-	0	0	20
10	-	75	0
20	25	-	0
0	10	15	-

Row scanning & column scanning method

-	0	0	20
10	-	75	0
20	25	-	0
0	10	15	-

undeleted(4)
detc (copy)
inter(4)
 $x=10$

iteration - I

-	0	0	30
10	-	65	0
20	15	-	0
0	0	5	-

iteration - II

	A	B	C	D
A	-	0	0	40
B	0	-	55	0
C	10	5	-	0
D	0	0	5	-

$A \rightarrow C \rightarrow D \rightarrow B \rightarrow A$

$$75 + 15 + 75 + 35 = ₹ 200$$

→ Method - the cell enters the processing time in hours of the job (i) if it is assigned to operator (j)

		operator (j)			
		1	2	3	4
job (i)	1	13	5	8	10
	2	9	15	18	10
	3	12	14	10	10
	4	10	14	9	12

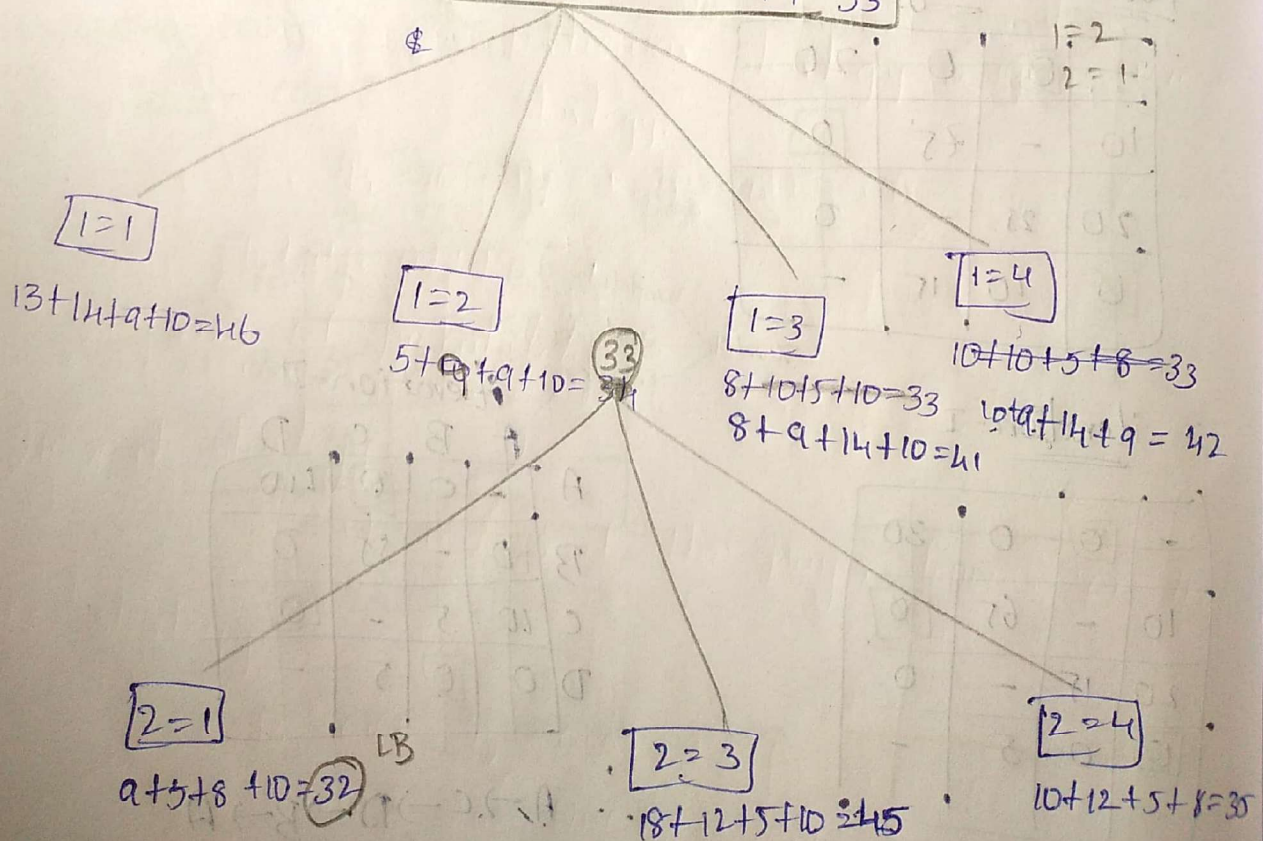
Sol Sol

Grillen

Assignment problem is balanced. Assignment problem.

NO. of rows = NO. of columns

Lower Bound = $5 + 9 + 10 + 9 = 33$



$$2=1$$

$$9+5+8+10=32$$

$$1=2, 2=1, 3=3, 4=4$$

$$5+9+10+12=36$$

$$1=2, 2=1, 3=4, 4=3$$

$$5+9+10+9=33$$

Job	operator	time
-----	----------	------

1	2	5
---	---	---

2	1	9
---	---	---

3	4	10
---	---	----

4	3	9
		33 UB

Lower bound

33 to

upper bound

33

Network Techniques1. Shortest-path mode

- (i) Systematic method
- (ii) Dijkstra's Algorithm
- (iii) Floyd's Algorithm

2. Minimum Spanning Tree model:

- (i) Prime Algorithm
- (ii) Kruskal's Algorithm

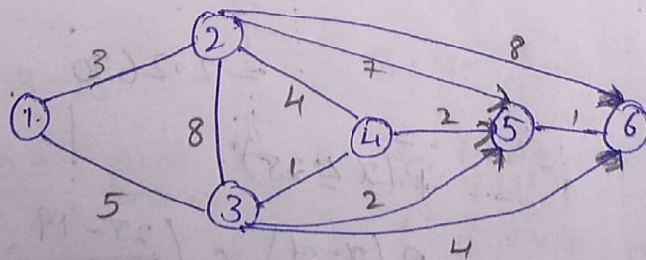
3. Maximal Flow problems:

- (i) Linear programming modeling for maximal flow problem
- (ii) maximal flow problem algorithm.

1. Shortest path mode:

Floyd's Algorithm: [It can find the shortest path]

1. consider the distance Nlw. shown in below.



→ Apply Floyd's algorithm to it & generate the final distance matrix.

Iteration-0

Distance matrix $[D^0]$

	1	2	3	4	5	6
1	0	3	5	∞	∞	∞
2	3	0	8	4	7	8
3	5	8	0	1	2	4
4	∞	4	1	0	2	∞
5	∞	∞	2	∞	0	1
6	∞	8	4	∞	1	0

Iteration-I

Distance matrix $[D^1]$

	1	2	3	4	5	6
1	0	3	5	∞	∞	∞
2	3	0	8	4	7	8
3	5	8	0	1	2	4
4	∞	4	1	0	2	∞
5	∞	∞	2	∞	0	1
6	∞	8	4	∞	1	0

$5+3=8$
 $8+0$
 $\Rightarrow (3+3, 0)$
 $(6, 0)$
 $\Rightarrow (3+5, 8)$
 $(8, 8)$
 $\Rightarrow (3+2, 4)$
 4
 $\Rightarrow (3+2, 7)$
 7

Iteration-II

Distance Matrix $[D^2]$

	1	2	3	4	5	6
1	0	3	5	7	10	11
2	3	0	8	4	7	8
3	5	8	0	1	2	4
4	7	4	1	0	2	12
5	∞	∞	2	∞	0	1
6	11	8	4	12	1	0

Iteration-III

distance matrix $[D^3]$

	1	2	3	4	5	6
1	0	3	5	6	7	9
2	3	0	8	4	7	8
3	5	8	0	1	2	4
4	6	4	1	0	2	5
5	7	10	2	3	0	1
6	9	8	4	5	1	0

Iteration-IV

distance matrix $[D^4]$

	1	2	3	4	5	6
1	0	3	5	6	7	9
2	3	0	5	4	6	8
3	5	5	0	1	2	4
4	6	4	1	0	2	5
5	7	7	2	3	0	1
6	9	8	4	5	1	0

Iteration-V

distance matrix $[D^5]$

	1	2	3	4	5	6
1	0	3	5	6	7	8
2	3	0	5	4	6	7
3	5	5	0	1	2	3
4	6	4	1	0	2	3
5	7	7	2	3	0	1
6	8	8	3	4	1	0

Iteration-VI

distance matrix $[D^6]$

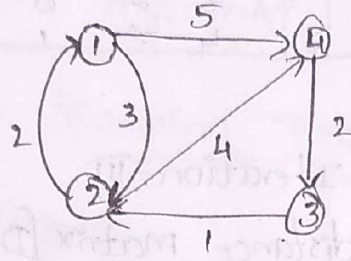
	1	2	3	4	5	6
1	0	3	5	6	7	8
2	3	0	5	4	6	7
3	5	5	0	1	2	3
4	6	4	1	0	2	3
5	7	7	2	3	0	1
6	8	8	3	4	1	0

2) Floyd's Algorithm:

Shortest path:

2) consider the distance NW-shown in the below.

Apply Floyd's algorithm to it & generate the final distance matrix.



80] Iteration [0]

distance matrix $[D^0]$

	1	2	3	4
1	0	3	∞	5
2	2	0	∞	4
3	∞	1	0	∞
4	∞	∞	2	0

Iteration [1]

distance matrix $[D^1]$

	1	2	3	4
1	0	3	∞	5
2	2	0	∞	4
3	∞	1	0	∞
4	∞	∞	2	0

Iteration [2]

distance matrix $[D^2]$

	1	2	3	4
1	0	3	∞	5
2	2	0	∞	4
3	3	1	0	5
4	∞	∞	2	0

Iteration [3]

distance matrix $[D^3]$

	1	2	3	4
1	0	3	∞	5
2	2	0	∞	4
3	3	1	0	5
4	5	3	2	0

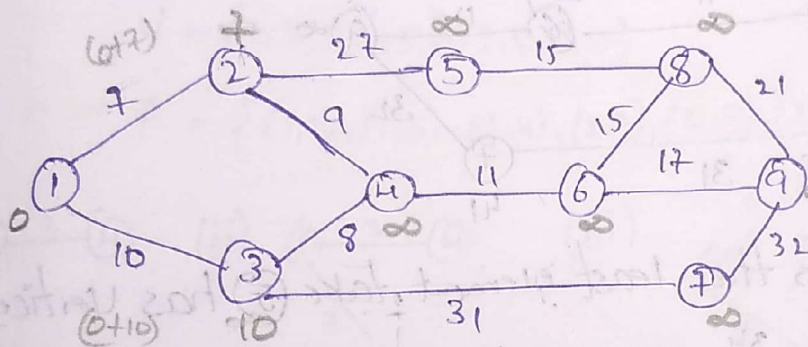
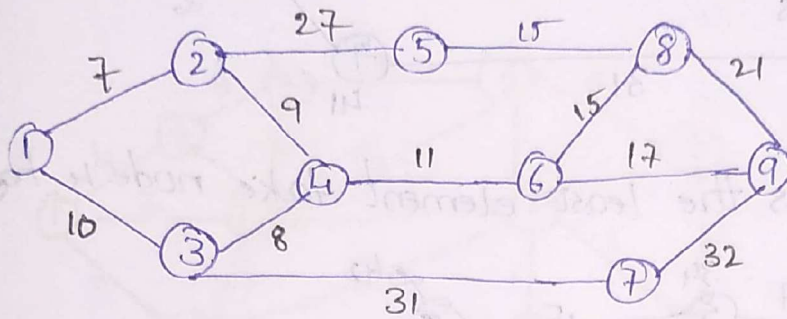
Iteration [4]

distance matrix $[D^4]$

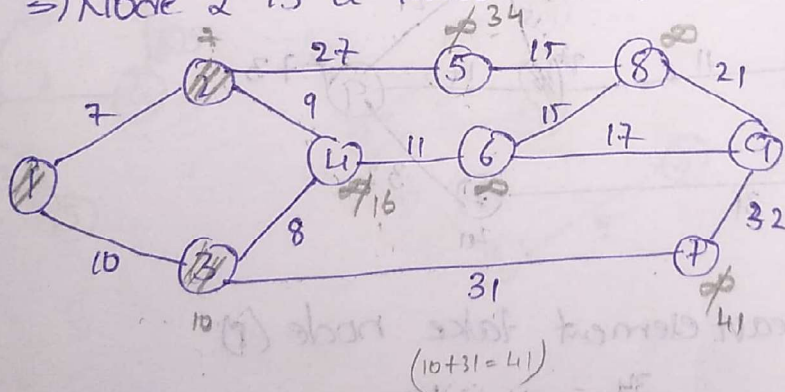
	1	2	3	4
1	0	3	7	5
2	2	0	6	4
3	3	1	0	5
4	5	3	2	0

Dijkstra's Algorithm:

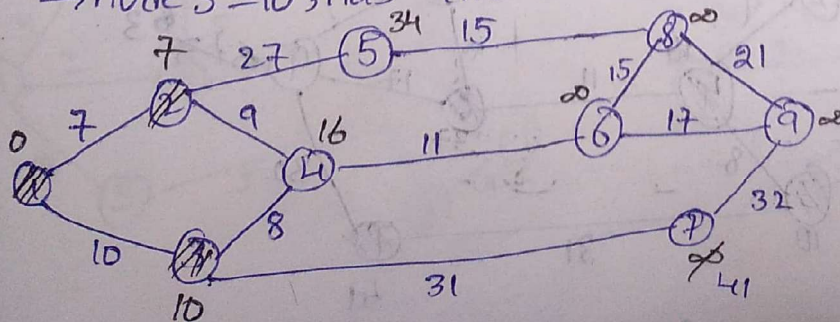
Find the shortest path from node 1 to node 9 of the distance M.W. show in the following using Dijkstra's Algorithm.



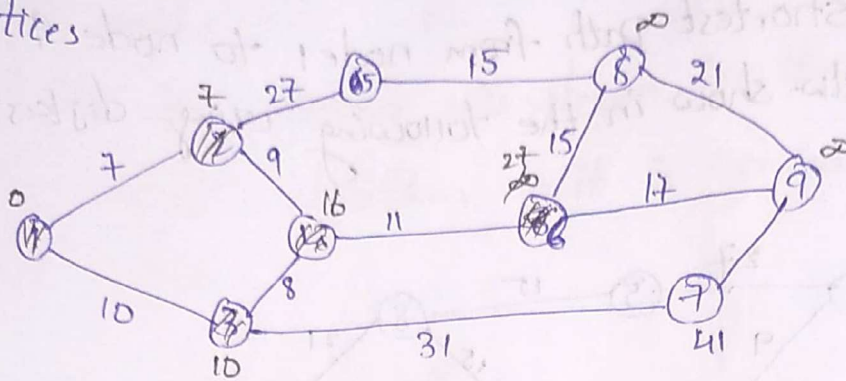
$\Rightarrow$ Node 2 is a least value, take node 2 has vertices.



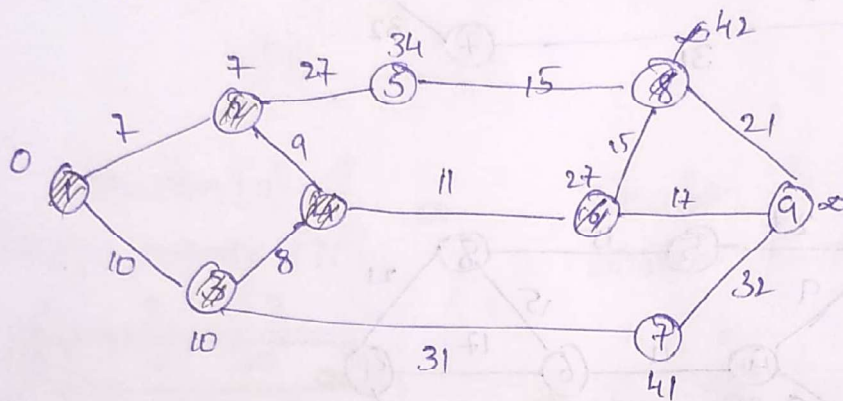
$\Rightarrow$ node 3 = 10, has the least element, take (3) has vertices



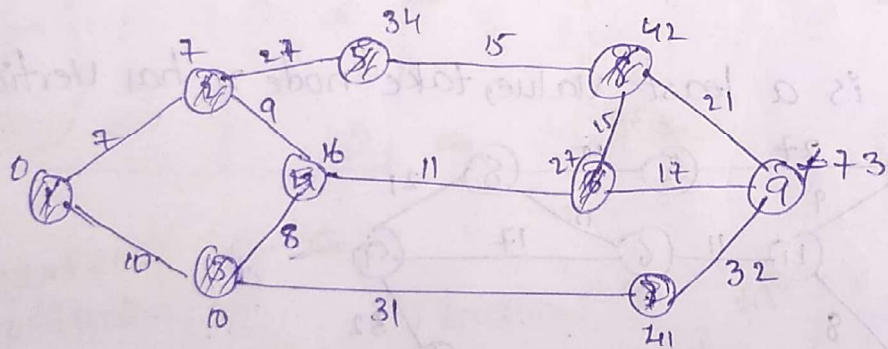
$\Rightarrow$ node 4 = 16, has the least element take node 4 has Vertices



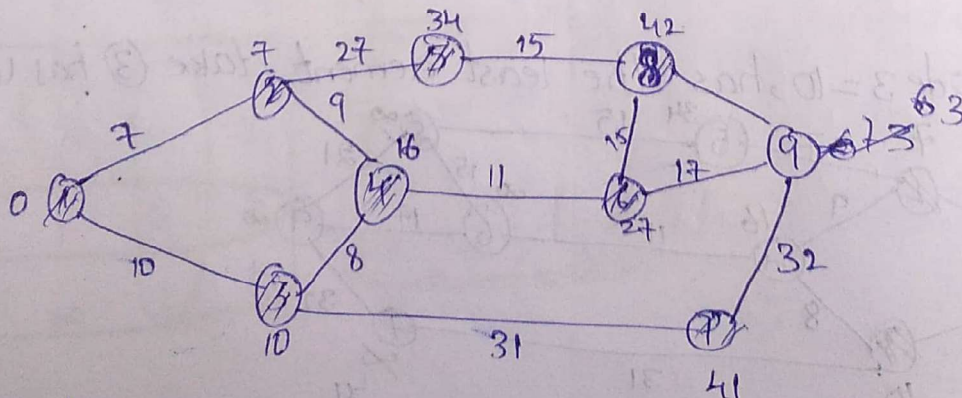
$\Rightarrow$ node 6 = 27, has the least element take node 6 has Vertices



$\Rightarrow$ node 5 = 34, has the least element, take 5 has Vertices



$\Rightarrow$ node 8 = 42, least element take node 8

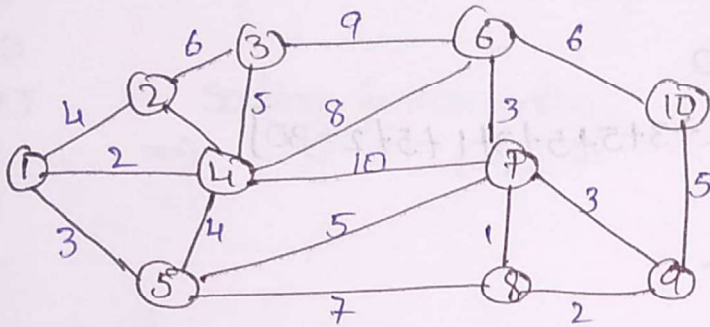


Shortest path:- 1-2-4-6-8-9

minimum spanning Tree models:-

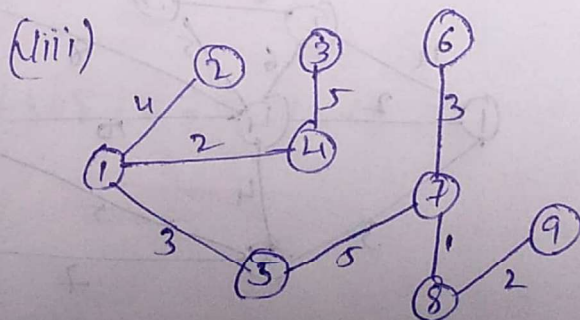
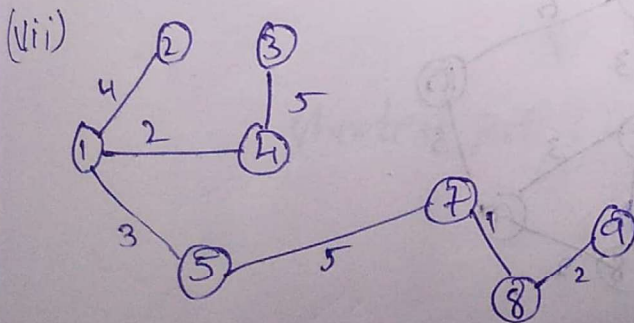
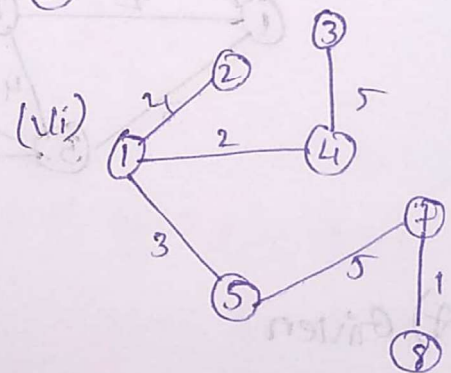
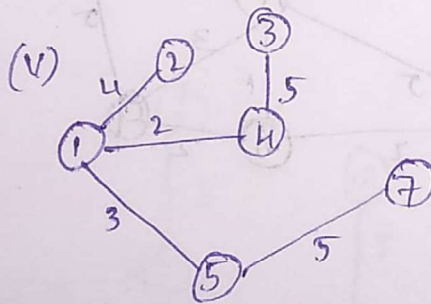
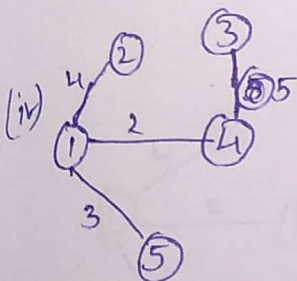
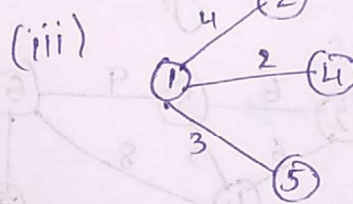
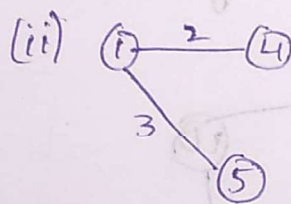
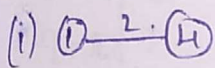
prim's Algorithm:-

1) find shortest path, consider the distance now shown in the below, find the minimum spanning tree of this N/w using prim's algorithm.

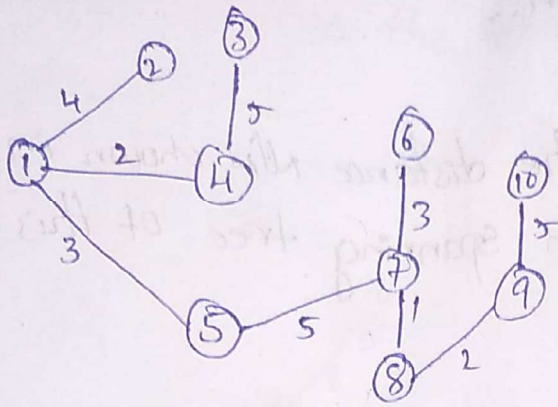


$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$E = \{(1,2), (1,5), (1,4), (2,3), (4,5), (2,4), \dots, (9,10)\}$$



(ix)



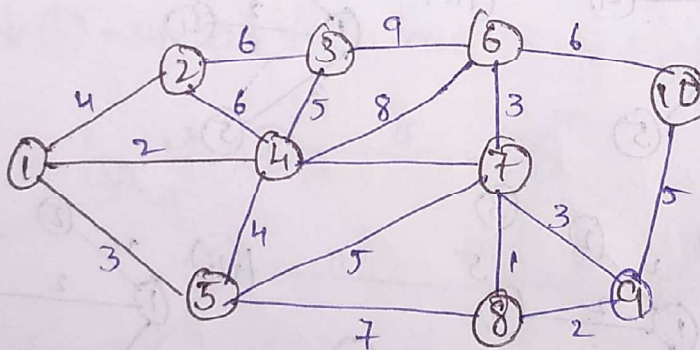
∴ Shortest path: 30

$$[4+2+3+5+5+3+1+5+2=30]$$

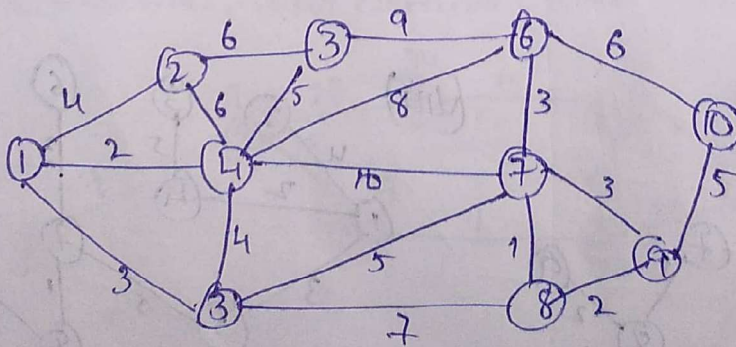
4) Floyd's Algorithm:-

Kruskal's Algorithm:-

1) consider the n/w given below. Find the minimum spanning tree using kruskal's algorithm.



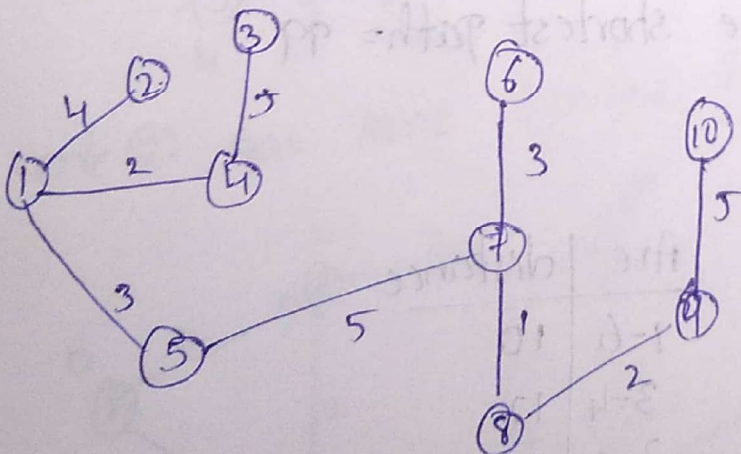
A) Given



Arc	distance
1-2	4
1-4	2
1-3	3
2-4	6
2-3	6
4-6	8
4-7	10
4-3	5
4-5	9
3-6	9
3-4	5
5-7	5
5-8	7
6-10	6
7-8	1
7-9	3
8-9	2
9-10	5

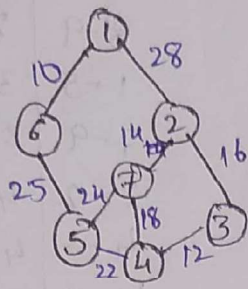
⇒ Sorting in ascending order

Arc	distance
7-8	1
1-4	2
8-9	2
1-5	3
7-9	3
1-2	4
4-5	4
4-3	5
3-4	5
5-7	5
9-10	5
2-4	6
2-3	6
6-10	6
5-8	7
4-6	8
3-6	9
4-7	10

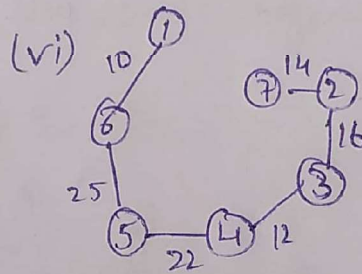
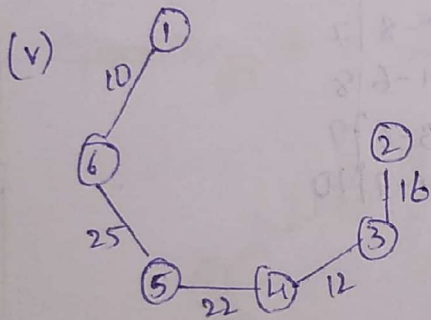
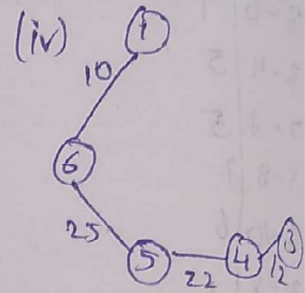
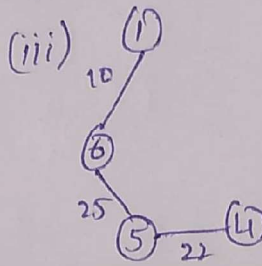
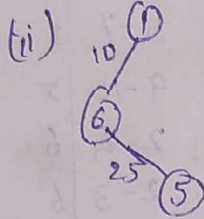
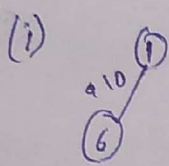


Shortest path: $4 + 2 + 3 + 5 + 5 + 3 + 1 + 2 + 5 = 30$

② find the shortest path, by using prim's algorithm & Kruskal's algorithm to the following graph.



A) prim's algorithm



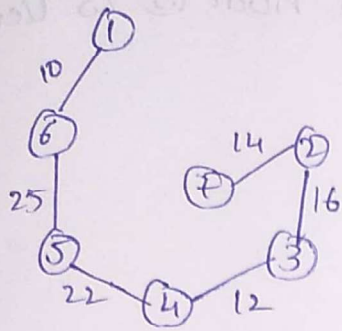
The shortest path = 99

* Kruskal's algorithm

Arc	distance
1-6	10
6-5	25
5-4	22
4-3	12
3-2	16
2-7	14
7-2	18
5-7	24
4-7	18
5-4	22
5-7	24
6-5	25
2-1	28

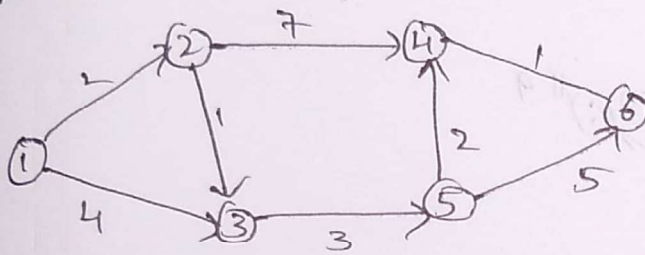
sorting
→

Arc	distance
1-6	10
3-4	12
7-2	14
2-3	16
4-7	18
5-4	22
5-7	24
6-5	25
2-1	28

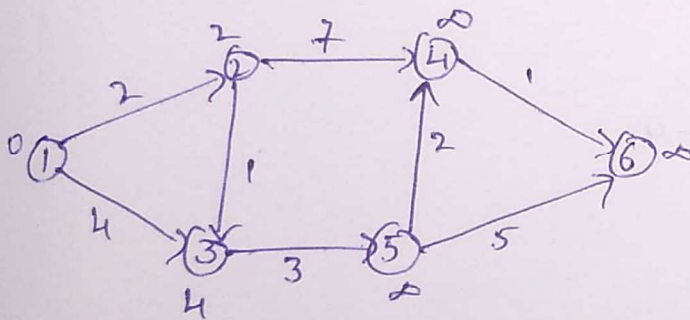


The shortest path = 99

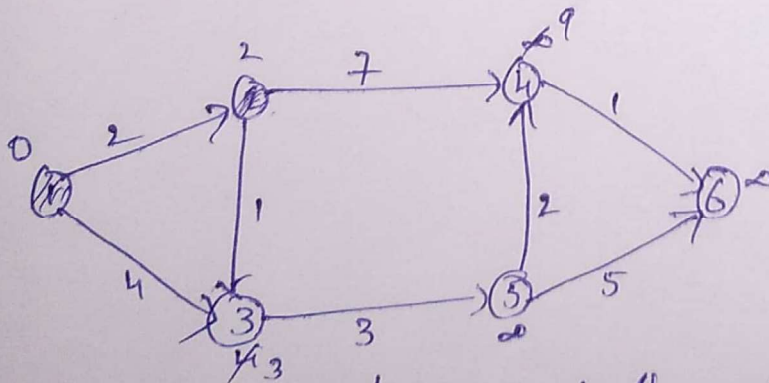
find the shortest path from node ① to node ⑥
at the distance network shown in the following
N/W



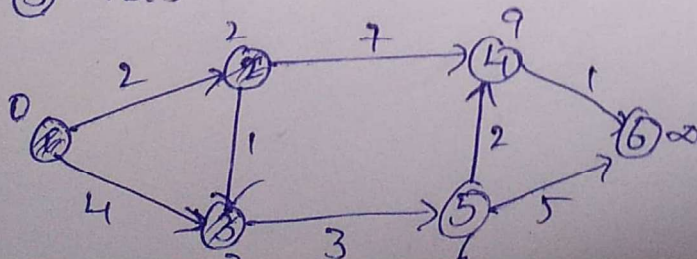
A) (i)



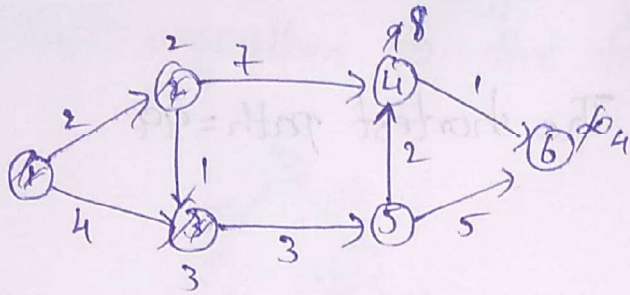
node ② has least element, then node ② has vertices



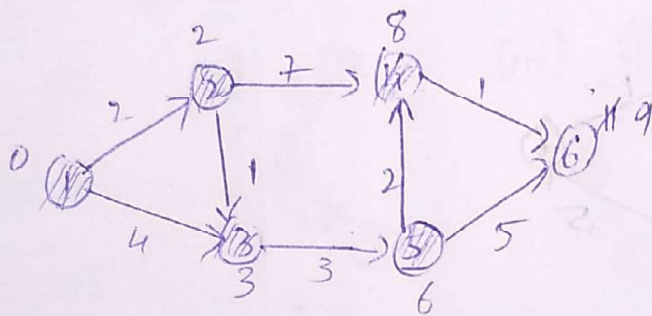
node ③ has least element, then node ③ has vertices



node ⑤, has the least element, then node ⑤ is vertex



node ④, has the least element, then node ④ is vertex



The shortest path is

1-2-4-5-4-6

$$2+1+3+2+1=9$$

Network Techniques

5.1 INTRODUCTION

A *network* consists of a set of nodes (vertices) and a set of arcs (edges). Each node represents a location (city) and each arc represents the connection (road link) between two different locations (cities). The number on each arc represents the distance between the two locations (cities) (refer to Fig. 5.1).

There are three types of network techniques:

1. Shortest-path model
2. Minimum spanning tree model
3. Maximal flow model.

The 'shortest-path model' can be further classified into two: shortest path between only one pair of nodes and shortest path between any pair of nodes in a network.

The shortest path between only one pair of nodes can be determined using a systematic method or Dijkstra's algorithm. The shortest path between any pair of nodes in a given network can be determined using Floyd's algorithm. The minimum spanning tree problem can be solved using either PRIM algorithm or Kruskal's algorithm.

5.2 SHORTEST-PATH MODEL

In transport organizations, one of the objectives is to find the shortest path to a particular node from any of the other nodes in a road network for shipping cargo. Determination of the shortest path using specialized procedures is known as *shortest-path model*. The following methods are used to find the shortest path in a distance network:

1. Systematic method
2. Dijkstra's algorithm
3. Floyd's algorithm

These are discussed in the next sections.

5.2.1 Systematic Method

The algorithm of systematic method for determining the shortest path between two given nodes is presented as follows:

- Step 1:** Represent the details of the distance network in the form of a table (say first table). For each of the nodes in the network, a column is provided in this table. The arcs that are emanating from a given node are arranged as per the increasing order of their distances and presented in the corresponding column of the first table from top to bottom.
- Step 2:** Select Node 1 and set the cumulative distance covered up to node 1 to 0 as shown at the top of the respective column of the first table.
- Step 3:** Delete all the arcs in the first table that are pointing towards node 1.
- Step 4:** Include the recently selected node in List A of another table (say, second table).
- Step 5:** Find the nearest node for each of the nodes in List A and write it in the third column of the second table. (If all the arcs are deleted in the first table corresponding to any node in the List A, then remove that node from the List A of the second table.)
- Step 6:** For each node in List A of the second table, calculate the cumulative distance up to its nearest node as shown in the last column of the second table. Then, select the nearest node which has the least cumulative distance (X) and mark a square around it in the last column of the second table. Write the cumulative distance covered up to the selected node as X at the top of the respective column of the first table.
- Step 7:** Check whether the recently selected node is same as the required destination. If so, go to Step 9; otherwise go to Step 8.
- Step 8:** Delete all the arcs in the first table, that are pointing towards the recently selected node, and then go to Step 4.
- Step 9:** Treat the lastly selected node as the last node in the shortest path.
- Step 10:** Find the node in List A corresponding to the recently selected/prefixed* node and prefix that node in the partially formed shortest path.
- Step 11:** Check whether the prefixed node is the required source node. If no, go to Step 12; otherwise to Step 13.
- Step 12:** Move to the iteration in which the recently prefixed node is selected (found with square) in the last column of the second table. Then, go to Step 10.
- Step 13:** Treat the path which is constructed, based on the above guidelines, as the shortest path and the shortest distance for this shortest path is equal to the least cumulative distance in the last iteration of the second table.

Example 5.1 Consider the road network as in Fig. 5.1, where distances between different pairs of adjacent cities are summarized. Find the shortest path from City 1 to City 10.

Solution

- Step 1:** The distance network which is shown in Fig. 5.1 is represented in matrix form as shown in Table 5.1.

*The term 'prefixed' will have its meaning while Step 10 is repeated.

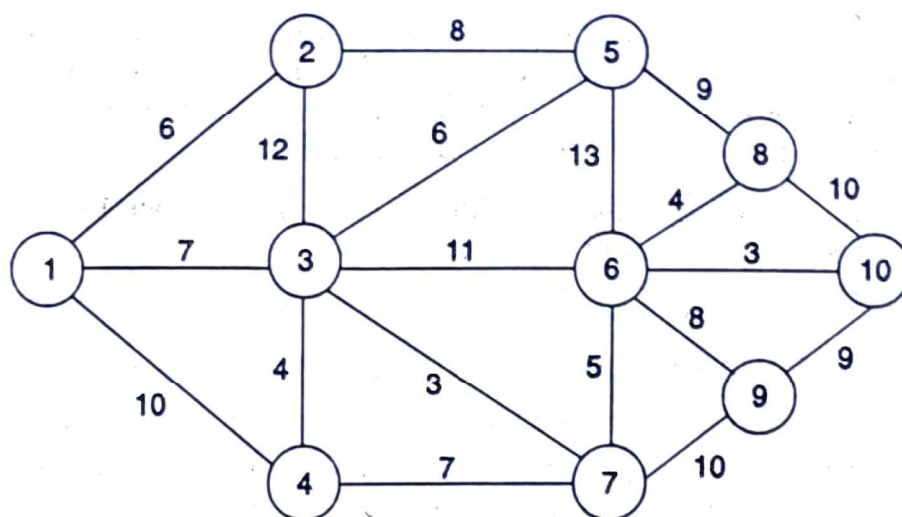


Fig. 5.1 Distance network.

Table 5.1 Details of Distances (Iteration 1)

1	2	3	4	5	6	7	8	9	10
1-2 6	2-1 6	3-7 3	4-3 4	5-3 6	6-10 3	7-3 3	8-6 4	9-6 8	10-6 3
1-3 7	2-5 8	3-4 4	4-7 7	5-2 8	6-8 4	7-6 5	8-5 9	9-10 9	10-9 9
1-4 10	2-3 12	3-5 6	4-1 10	5-8 9	6-7 5	7-4 7	8-10 10	9-7 10	10-8 10
		3-1 7		5-6 13	6-9 8	7-9 10			
		3-6 11			6-3 11				
		3-2 12			6-5 13				

Step 2: City 1 is selected as shown in Table 5.2, and the cumulative distance travelled up to City 1 is set to 0 as shown in the last column of Table 5.2 as well as at the top of the column 1 of Table 5.3 of the next step.

Table 5.2 Details of Selection of Node in Iteration 1

Iteration	List A (nodes included)	Nearest nodes	Distance calculation
1	-	1	0

Step 3: Delete all the arcs in Table 5.1 that are pointing towards City 1 (2-1, 3-1 and 4-1) as shown in Table 5.3 (deletion of arc is indicated using a '*' by the side of that arc).

Table 5.3 Updated Distance (Iteration 2)

0	1	2	3	4	5	6	7	8	9	10
1-2 6	2-1 6*	3-7 3	4-3 4	5-3 6	6-10 3	7-3 3	8-6 4	9-6 8	10-6 3	
1-3 7	2-5 8	3-4 4	4-7 7	5-2 8	6-8 4	7-6 5	8-5 9	9-10 9	10-9 9	
1-4 10	2-3 12	3-5 6	4-1 10*	5-8 9	6-7 5	7-4 7	8-10 10	9-7 10	10-8 10	
		3-1 7*		5-6 13	6-9 8	7-9 10				
		3-6 11			6-3 11					
		3-2 12			6-5 13					

Steps 4, 5 and 6: The calculations and updations with respect to these steps are shown in Table 5.4.

Table 5.4 Details of Selection of Node in Iteration 2

Iteration	List A (nodes included)	Nearest nodes	Distance calculation
1	—	1	0
2	1	2	$0 + 6 = 6^*$

Step 7: The recently selected City 2 is not the required City 10. So, go to Step 8.

Step 8: Delete all the arcs that are pointing towards City 2 (1-2, 3-2 and 5-2) as shown in Table 5.5, and then go to Step 4.

Table 5.5 Updated Distance (Iteration 3)

0	6																		
1	2	3	4	5	6	7	8	9	10										
1-2 6*	2-1 6*	3-7 3	4-3 4	5-3 6	6-10 3	7-3 3	8-6 4	9-6 8	10-6 3										
1-3 7	2-5 8	3-4 4	4-7 7	5-2 8*	6-8 4	7-6 5	8-5 9	9-10 9	10-9 9										
1-4 10	2-3 12	3-5 6	4-1 10*	5-8 9	6-7 5	7-4 7	8-10 10	9-7 10	10-8 10										
		3-1 7*		5-6 13	6-9 8	7-9 10													
		3-6 11			6-3 11														
		3-2 12*			6-5 13														

Step 4, 5 and 6: The calculations and updations with respect to these steps are shown in Table 5.6.

Table 5.6 Details of Selection of Node in Iteration 3

Iteration	List A (nodes included)	Nearest nodes	Distance calculation
1	—	1	0
2	1	2	$0 + 6 = 6^*$
3	1, 2	3 , 5	$0 + 7 = 7^*$ $6 + 8 = 14$

Step 7: The recently selected city, i.e. City 3 is not the required City 10. So, go to Step 8.

Step 8: Delete all the arcs that are pointing towards City 3 (1-3, 2-3, 5-3, 6-3, 7-3 and 4-3) as shown in Table 5.7 and then go to Step 4.

Table 5.7 Updated Distance (Iteration 4)

0	6	7																	
1	2	3	4	5	6	7	8	9	10										
1-2	6*	2-1	6*	3-7	3	4-3	4*	5-3	6*	6-10	3	7-3	3*	8-6	4	9-6	8	10-6	3
1-3	7*	2-5	8	3-4	4	4-7	7	5-2	8*	6-8	4	7-6	5	8-5	9	9-10	9	10-9	9
1-4	10	2-3	12*	3-5	6	4-1	10*	5-8	9	6-7	5	7-4	7	8-10	10	9-7	10	10-8	10
			3-1	7*				5-6	13	6-9	8	7-9	10						
			3-6	11						6-3	11*								
			3-2	12*						6-5	13								

Steps 4, 5 and 6: The calculations and updations with respect to these steps are shown in Table 5.8.

Table 5.8 Details of Selection of Node in Iteration 4

Iteration	List A (nodes included)	Nearest nodes	Distance calculation
1	—	1	0
2	1	2	$0 + 6 = 6^*$
3	1, 2	3 , 5	$0 + 7 = 7^*$ $6 + 8 = 14$
4	1, 2, 3	4, 5, 7	$0 + 10 = 10^*$ $6 + 8 = 14$ $7 + 3 = 10^*$

In Table 5.8, City 7 is selected by breaking tie between City 4 and City 7, randomly.

Step 7: The recently selected City 7 is not the required City 10. So, go to Step 8.

Step 8: Delete all the arcs that are pointing towards City 7 (4-7, 3-7, 6-7 and 9-7) as shown in Table 5.9 and then go to Step 4.

Table 5.9 Updated Distance (Iteration 5)

0	6	7	10									
1	2	3	4	5	6	7	8	9	10			
1-2 6*	2-1 6*	3-7 3*	4-3 4*	5-3 6*	6-10 3	7-3 3*	8-6 4	9-6 8	10-6 3			
1-3 7*	2-5 8	3-4 4	4-7 7*	5-2 8*	6-8 4	7-6 5	8-5 9	9-10 9	10-9 9			
1-4 10	2-3 12*	3-5 6	4-1 10*	5-8 9	6-7 5*	7-4 7	8-10 10	9-7 10*	10-8 10			
		3-1 7*		5-6 13	6-9 8	7-9 10						
		3-6 11			6-3 11*							
		3-2 12*			6-5 13							

Steps 4, 5 and 6: The calculations and updations with respect to these steps are shown in Table 5.10.

Table 5.10 Details of Selection of Node in Iteration 5

Iteration	List A (nodes included)	Nearest nodes	Distance calculation
1	-	1	0
2	1	2	$0 + 6 = 6^*$
3	1, 2	3 , 5	$0 + 7 = 7^*$ $6 + 8 = 14$
4	1, 2, 3	4, 5, 7	$0 + 10 = 10^*$ $6 + 8 = 14$ $7 + 3 = 10^*$
5	1, 2, 3, 7	4 , 5, 4, 6	$0 + 10 = 10^*$ $6 + 8 = 14$ $7 + 4 = 11$ $10 + 5 = 15$

Step 7: City 4, that is selected recently, is not the required City 10. So, go to Step 8.

Step 8: Delete all the arcs that are pointing towards City 4 (1-4, 3-4 and 7-4) as shown in Table 5.11 and then go to Step 4.

Table 5.11 Updated Distance (Iteration 6)

0	6	7	10	10					
1	2	3	4	5	6	7	8	9	10
1-2 6*	2-1 6*	3-7 3*	4-3 4*	5-3 6*	6-10 3	7-3 3*	8-6 4	9-6 8	10-6 3
1-3 7*	2-5 8	3-4 4*	4-7 7*	5-2 8*	6-8 4	7-6 5	8-5 9	9-10 9	10-9 9
1-4 10*	2-3 12*	3-5 6	4-1 10*	5-8 9	6-7 5*	7-4 7*	8-10 10	9-7 10*	10-8 10
		3-1 7*		5-6 13	6-9 8	7-9 10			
		3-6 11			6-3 11*				
		3-2 12*			6-5 13				

Steps 4, 5 and 6: The calculations and updations with respect to these steps are shown in Table 5.12.

In Table 5.11, all the arcs with respect to Node 1 and Node 4 are deleted. Hence, in Table 5.12 under List A, these nodes are deleted by marking 'X' against each of them. The cumulative distance up to the nearest neighbour of each of the remaining nodes, i.e. 2, 3 and 7 are shown in the last row of Table 5.12.

Table 5.12 Details of Selection of Node in Iteration 6

Iteration	List A (nodes included)	Nearest nodes	Distance calculation
1	—	1	0
2	1	2	$0 + 6 = 6^*$
3	1, 2	3 , 5	$0 + 7 = 7^*$ $6 + 8 = 14$
4	1, 2, 3	4, 5, 7	$0 + 10 = 10^*$ $6 + 8 = 14$ $7 + 3 = 10^*$
5	1, 2, 3, 7	4 , 5, 4, 6	$0 + 10 = 10^*$ $6 + 8 = 14$ $7 + 4 = 11$ $10 + 5 = 15$
	X X		
6	1, 2, 3, 7, 4	5, 5 , 6	$6 + 8 = 14$ $7 + 6 = 13^*$ $10 + 5 = 15$

Step 7: The recently selected City 5 is not the required destination City. So, go to Step 8.

Step 8: Delete all the arcs that are pointing towards City 5 (2–5, 3–5, 6–5 and 8–5) as shown in Table 5.13 and then go to Step 4.

Table 5.13 Updated Distance (Iteration 7)

0	6	7	10	13				10		
1	2	3	4	5	6	7	8	9	10	
1–2 6*	2–1 6*	3–7 3*	4–3 4*	5–3 6*	6–10 3	7–3 3*	8–6 4	9–6 8	10–6 3	
1–3 7*	2–5 8*	3–4 4*	4–7 7*	5–2 8*	6–8 4	7–6 5	8–5 9*	9–10 9	10–9 9	
1–4 10*	2–3 12*	3–5 6*	4–1 10*	5–8 9	6–7 5*	7–4 7*	8–10 10	9–7 10*	10–8 10	
		3–1 7*		5–6 13	6–9 8	7–9 10				
		3–6 11			6–3 11*					
		3–2 12*			6–5 13*					

Steps 4, 5 and 6: The calculations and updates with respect to these steps are shown in Table 5.14.

Step 7: The recently selected City 6 is not the required City 10. So, go to Step 8.

Step 8: Delete all the arcs that are pointing towards City 6 (3–6, 5–6, 8–6, 10–6, 9–6 and 7–6) as shown in Table 5.15, and then go to Step 4.

Table 5.14 Details of Selection of Node in Iteration 7

Iteration	List A (nodes included)	Nearest nodes	Distance calculation
1	—	1	0
2	1	2	$0 + 6 = 6^*$
3	1, 2	3 , 5	$0 + 7 = 7^*$ $6 + 8 = 14$
4	1, 2, 3	4, 5, 7	$0 + 10 = 10^*$ $6 + 8 = 14$ $7 + 3 = 10^*$
5	1, 2, 3, 7	4 , 5, 4, 6	$0 + 10 = 10^*$ $6 + 8 = 14$ $7 + 4 = 11$ $10 + 5 = 15$ $6 + 8 = 14$
	X X		
6	1, 2, 3, 7, 4	5, 5 , 6	$7 + 6 = 13^*$ $10 + 5 = 15$
	X		
7	2, 3, 7, 5	6, 6 , 8	$7 + 11 = 18$ $10 + 5 = 15^*$ $13 + 9 = 22$

Table 5.15 Updated Distance (Iteration 8)

0	6	7	10	13	15	10			
1	2	3	4	5	6	7	8	9	10
1-2 6*	2-1 6*	3-7 3*	4-3 4*	5-3 6*	6-10 3	7-3 3*	8-6 4*	9-6 8*	10-6 3*
1-3 7*	2-5 8*	3-4 4*	4-7 7*	5-2 8*	6-8 4	7-6 5*	8-5 9*	9-10 9	10-9 9
1-4 10*	2-3 12*	3-5 6*	4-1 10*	5-8 9	6-7 5*	7-4 7*	8-10 10	9-7 10*	10-8 10
	3-1 7*			5-6 13*	6-9 8	7-9 10			
	3-6 11*				6-3 11*				
	3-2 12*				6-5 13*				

Steps 4, 5 and 6: The calculations and updations with respect to these steps are shown in Table 5.16.

Table 5.16 Details of Selection of Node in Iteration 8

Iteration	List A (Nodes included)	Nearest nodes	Distance calculation
1	—	1	0
2	1	2	$0 + 6 = 6^*$
3	1, 2	3 , 5	$0 + 7 = 7^*$ $6 + 8 = 14$
4	1, 2, 3	4, 5, 7	$0 + 10 = 10^*$ $6 + 8 = 14$ $7 + 3 = 10^*$
5	1, 2, 3, 7	4 , 5, 4, 6	$0 + 10 = 10^*$ $6 + 8 = 14$ $7 + 4 = 11$ $10 + 5 = 15$
	X X		
6	1, 2, 3, 7, 4	5, 5 , 6	$6 + 8 = 14$ $7 + 6 = 13^*$ $10 + 5 = 15$
	X		
7	2, 3, 7, 5	6, 6 , 8	$7 + 11 = 18$ $10 + 5 = 15^*$ $13 + 9 = 22$
	X		
8	3, 7, 5, 6	9, 8, 10	$10 + 10 = 20$ $13 + 9 = 22$ $15 + 3 = 18^*$

[Note: The calculations of each of the two types of tables are shown in different tables mainly for better understanding. But each of them can be shown in a single table while solving a problem.]

Step 7: The recently selected City 10 is the required destination city. So, go to Step 9.

Step 9: The partial path: 10.

Step 10: The city in List A, corresponding to City 10, is 6. So, prefix City 6 in the partial path. The partial path is 6–10.

Step 11: The recently prefixed city is not the source city. So, go to Step 12.

Step 12: Move to Iteration 7 where City 6 is found with square and go to Step 10.

Step 10: The city in List A, corresponding to the City 6, is 7, So, prefix City 7 in the partial path. The partial path is 7–6–10.

Step 11: The recently prefixed city is not the source city. So, go to Step 12.

Step 12: Move to Iteration 4 where City 7 is found with square and go to Step 10.

Step 10: The city in the List A corresponding to the City 7, is 3, So, prefix City 3 in the partial path. Now, the partial path is 3-7-6-10.

Step 11: The recently prefixed city is not the source city. So, go to Step 12.

Step 12: Move to Iteration 3 where City 3 is found with square and go to Step 10.

Step 10: The city in the List A, corresponding to City 3, is 1. So, prefix City 1 in the partial path. Now, the partial path is 1-3-7-6-10.

Step 11: The recently prefixed city is the source city. So, go to Step 13.

Step 13: The shortest path, presented in Fig. 5.2 with double lines, covers a distance of 18 units. The root followed from City 1 is 1-3-7-6-10.

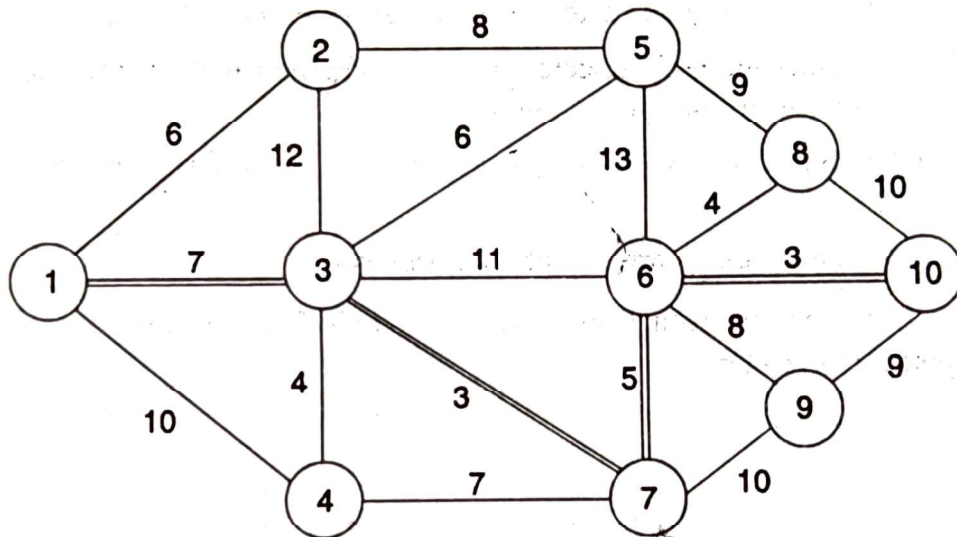


Fig. 5.2 Sketch of shortest path indicated using double line.

5.2.2 Dijkstra's Algorithm

Dijkstra's algorithm is another procedure to find the shortest path between any two nodes in a distance network. The arcs in the network may be directed or undirected. The steps of the algorithm are summarized below.

Step 1: Form a distance matrix from the start node to all other nodes. (If there is no direct arc from the start node to another node, the corresponding distance is assumed as ∞ .) Let the currently selected node (start node) be K .

Step 2: Find the smallest of the distances from the Node K to all other unselected nodes in the distance matrix. Let it be X and the corresponding node be L . Check whether L is a neighbour of the start node. If yes, do Step 2.1; otherwise, do Step 2.2.

2.1 Thicken (select) the arc connecting the start node and L . Then, go to Step 3.

2.2 Find the iteration among the preceding iterations in which the distance to L from the start node is different from X . Thicken (select) the arc connecting the selected node of that iteration and the selected node L of the present iteration. Then, go to Step 3.

Step 3: Check whether L is the required destination. If not, go to Step 4; otherwise, go to Step 6.

Step 4: Form the distance matrix by following the guidelines presented below. For each of the unselected neighbours of L , find its new distance by adding X to its distance from L , and update its old distance from the start node in the distance matrix if the new distance is smaller than its old distance. Transfer the data of the last node of the distance matrix to the column corresponding to the recently selected node, L . (This reduces the matrix column size by 1 in every iteration.)

Step 5: set $K = L$ and go to Step 2.

Step 6: Trace the shortest path.

6.1 Fix the selected node of the last iteration as the last node of the shortest path.

6.2 Traverse back through iterations and identify the iteration (j th iteration) at which the distance to the recently selected node is different.

6.3 Prefix the selected node of the j th iteration in the partially formed shortest path.

6.4 Check whether the node selected in the j th iteration is the source node. If not so, go to Step 6.5; otherwise, go to Step 6.6.

6.5 Treat the selected node in the j th iteration as the recently selected node and go to Step 6.2.

6.6 Construction of the shortest path is complete.

Example 5.2 Find the shortest path from Node 1 to Node 9 of the distance network shown in Fig. 5.3 using Dijkstra's algorithm.

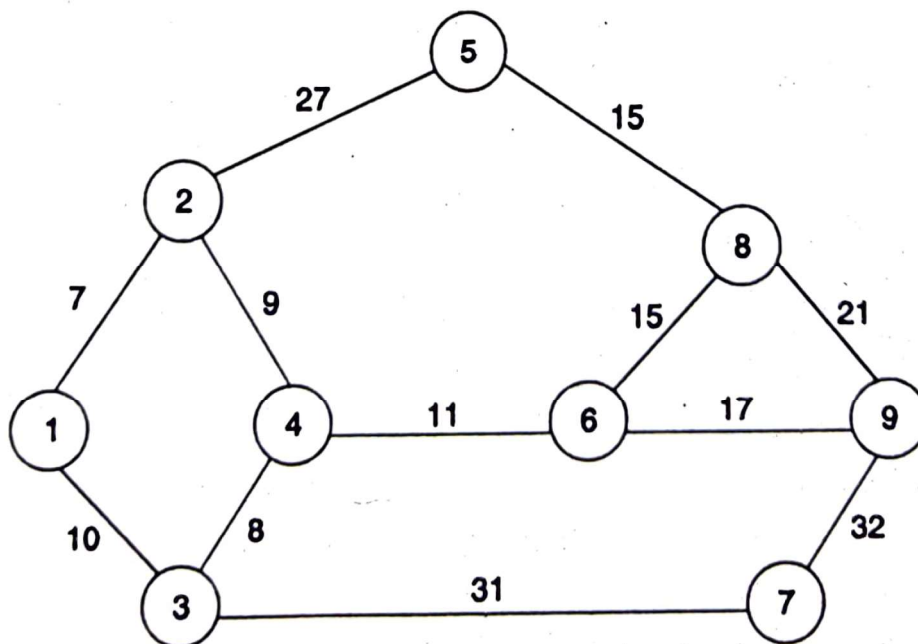


Fig. 5.3 Distance network for Example 5.2.

Solution

Iteration 1

Step 1: The distance matrix summarizing the distances from the start Node 1 to all other nodes

given in Fig. 5.4. In this figure one can see the actual distance from the start Node 1 to all its neighbours. The distance for all other nodes from Node 1 is assumed as ∞ .

2	3	4	5	6	7	8	9
7	10	∞	∞	∞	∞	∞	∞

Fig. 5.4 Distance matrix of Iteration 1.

Step 2: In Fig. 5.4, the smallest distance (X) is 7 and the corresponding node (L) is 2. Since the Node 2 is a neighbour of the start node, thicken the arc 1-2 as shown in Fig. 5.5.

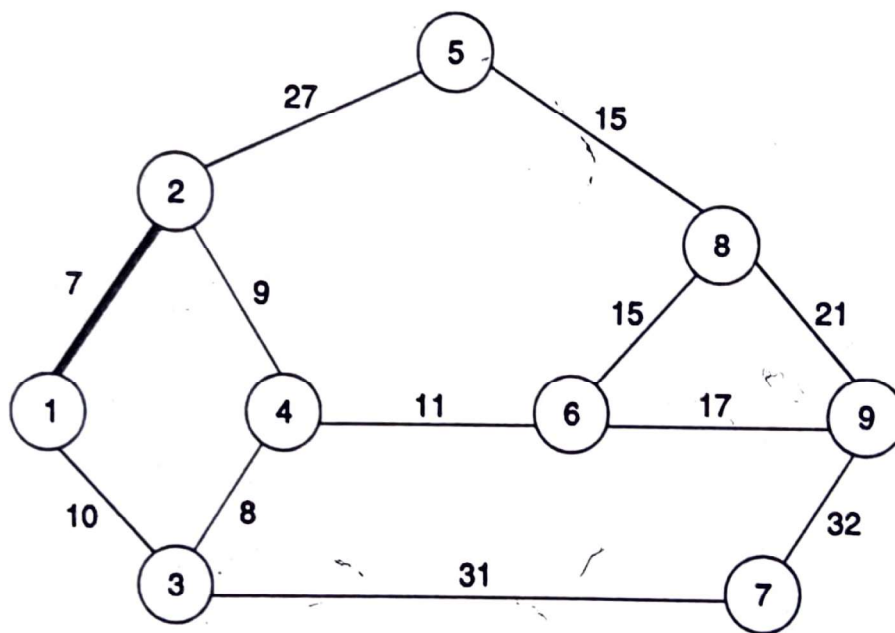


Fig. 5.5 Network showing partial solution.

Step 3: Since Node 2 is not the required destination, go to Step 4.

Iteration 2

Step 4: For Node 2 (L), Nodes 4 and 5 are the unselected neighbours. Then $X + d_{24} = 7 + 9 = 16$, where d_{24} is the actual distance from Node 2 to Node 4. Since this distance is less than ∞ (old distance up to Node 4 in Fig. 5.4), update the distance to Node 4 as 16. Then

$$X + d_{25} = 7 + 27 = 34$$

Again, since this distance is less than ∞ (old distance up to Node 5 in Fig. 5.4), update the distance to Node 5 as 34. Transfer the data of Node 9 to the position of Node 2. These are summarized in Fig. 5.6.

9	3	4	5	6	7	8
∞	10	16	34	∞	∞	∞

Fig. 5.6 Distance matrix of Iteration 2.

Step 5: Set $K = 2$ and go to Step 2.

Step 2: In Fig. 5.6, the smallest distance (X) is 10, and the corresponding node (L) is 3. Since Node 3 is a neighbour of the start node, thicken the arc 1-3 as shown in Fig. 5.7.

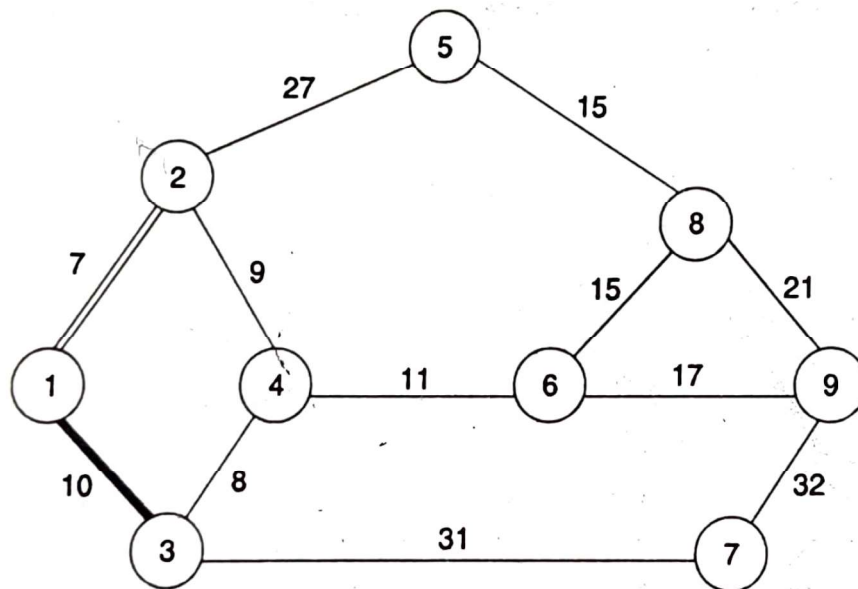


Fig. 5.7 Network showing partial solution.

Step 3: Since, the Node 3 is not the required destination, go to Step 4.

Iteration 3

Step 4: The results of this step are summarized as in Fig. 5.8.

9	8	4	5	6	7
∞	∞	16	34	∞	41

Fig. 5.8 Distance matrix of Iteration 3.

Step 5: Set $K = 3$ and go to Step 2.

Step 2: In Fig. 5.8, the smallest distance (X) is 16 and the corresponding node (L) is 4. Since, L is not a neighbour of the start node, perform Step 2.2.

2.2 The iteration among the preceding iterations in which the distance to L from the starting node is different from X is the Iteration 1. The node selected in that iteration is 2. So, thicken the arc 2-4 as shown in Fig. 5.9.

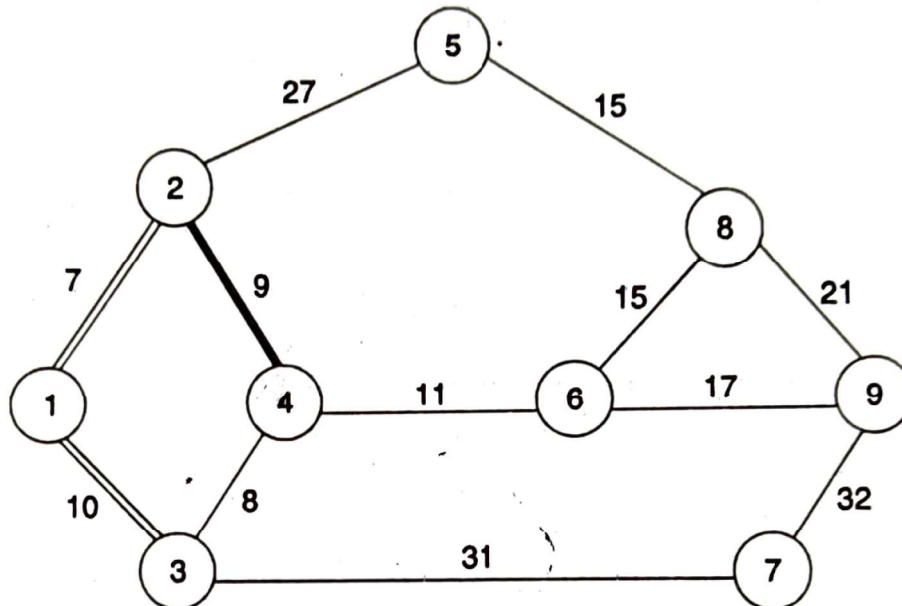


Fig. 5.9 Network showing partial solution.

Step 3: Since Node 4 is not the required destination, go to Step 4.

Iteration 4

Step 4: The results of this step are summarized as in Fig. 5.10.

9	8	7	5	6
∞	∞	41	34	27

Fig. 5.10 Distance matrix of Iteration 4.

Step 5: Set $K = 4$ and go to Step 2.

Step 2: In Fig. 5.10, the smallest distance (X) is 27 and the corresponding node (L) is 6. Since, L is not a neighbour of the start node, perform the Step 2.2.

2.2 The iteration among the preceding iterations in which the distance to L from the starting node is different from X is the Iteration 3. The node selected in that iteration is 4. So, thicken the arc 4-6 as shown in Fig. 5.11.

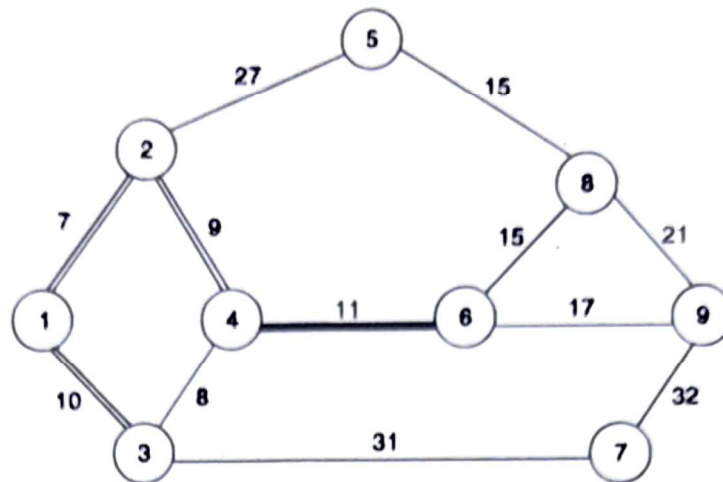


Fig. 5.11 Network showing partial solution.

Step 3: Since, Node 6 is not the required destination, go to Step 4.

Iteration 5

Step 4: The results of this step are summarized as in Fig. 5.12.

9	8	7	5
44	42	41	34

Fig. 5.12 Distance matrix of Iteration 5.

Step 5: Set $K = 6$ and go to Step 2.

Step 2: In Fig. 5.12, the smallest distance (X) is 34 and the corresponding node (L) is 5. Since L is not a neighbour of the start node, perform Step 2.2.

2.2 The iteration among the preceding iterations in which the distance to L from the starting node is different from X is the Iteration 1. The node selected in that iteration is 2. So, thicken the arc 2-5 as shown in Fig. 5.13.

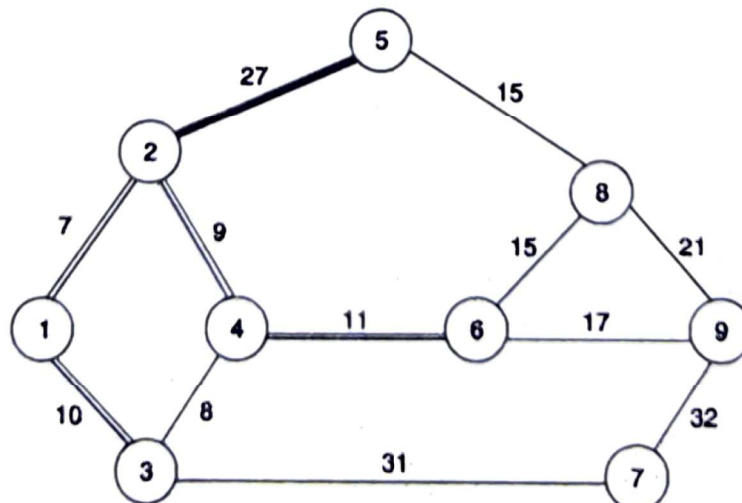


Fig. 5.13 Network showing partial solution.

Step 3: Since Node 5 is not the required destination, go to Step 4.

Iteration 6

Step 4: The results of this step are summarized as in Fig. 5.14.

9	8	7
44	42	41

Fig. 5.14 Distance matrix of Iteration 6.

Step 5: Set $K = 5$ and go to Step 2.

Step 2: In Fig. 5.14, the smallest distance (X) is 41 and the corresponding node (L) is 7. Since L is not a neighbour of the start node, perform Step 2.2.

2.2 The iteration among the preceding iterations in which the distance to L from the starting node is different from X is the Iteration 2. The node selected in that iteration is 3. So, thicken the arc 3-7 as shown in Fig. 5.15.

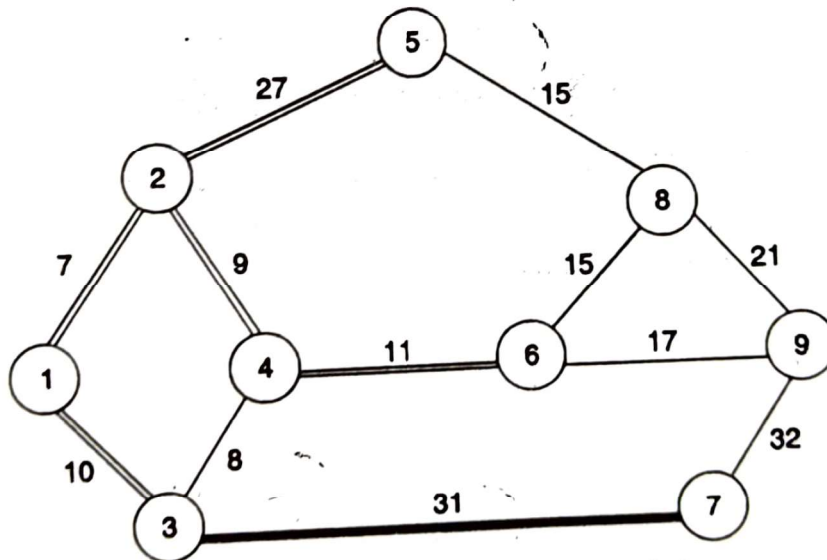


Fig. 5.15 Network showing partial solution.

Step 3: Since Node 7 is not the required destination, go to Step 4.

Iteration 7

Step 4: The results of this step are summarized as in Fig. 5.16.

9	8
44	42

Fig. 5.16 Distance matrix of Iteration 7.

Step 5: Set $K = 7$ and go to Step 2.

Step 2: In the Fig. 5.16, the smallest distance (X) is 42 and the corresponding node (L) is 8. Since, L is not a neighbour of the start node, perform Step 2.2.

2.2 The iteration among the preceding iterations in which the distance to L from the starting node is different from X is the Iteration 4. The node selected in that iteration is 6. So, thicken the arc 6–8 as shown in Fig. 5.17.

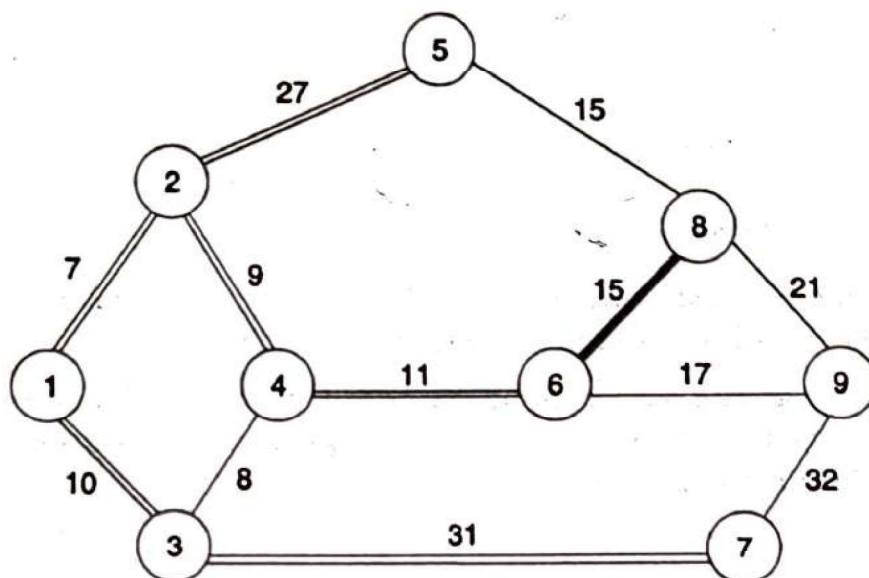


Fig. 5.17 Network showing partial solution.

Step 3: Since Node 8 is not the required destination, go to Step 4.

Iteration 8

Step 4: The results of this step are summarized in Fig. 5.18.



Fig. 5.18 Distance matrix of Iteration 8.

Step 5: Set $K = 8$ and go to Step 2.

Step 2: In the Fig. 5.18, the smallest distance (X) is 44 and the corresponding node (L) is 9. Since, L is not a neighbour of the start node, perform Step 2.2.

2.2 The iteration among the preceding iterations in which the distance to L from the starting node is different from X is Iteration 4. The node selected in that iteration is 6. So, thicken the arc 6–9 as shown in Fig. 5.19.

Step 3: Since Node 9 is the required destination, go to Step 6.

Step 6: Based on the guidelines of this step, the shortest path is 1–2–4–6–9 and the corresponding distance is 44.

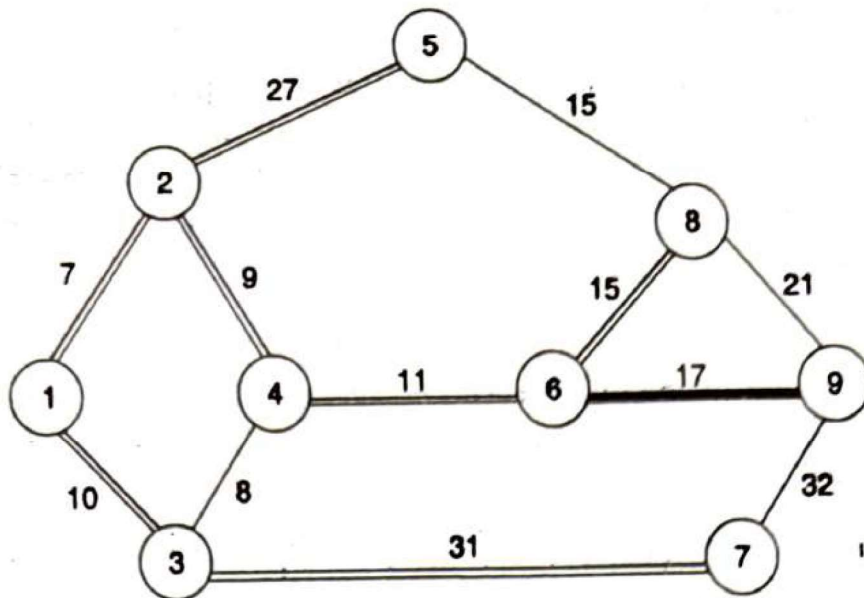


Fig. 5.19 Network showing final solution.

5.2.3 Floyd's Algorithm

Floyd's algorithm is used to find the shortest path and the corresponding distance from any source node to any destination node in a given distance network.

This algorithm takes the initial distance matrix $[D^0]$ and initial precedence matrix $[P^0]$ as input. Then, it performs n iterations (n is the number of nodes in the distance matrix) and generates the final distance matrix $[D^n]$ and the final precedence matrix $[P^n]$. One can find the shortest distance between any two nodes from the final distance matrix $[D^n]$ and can trace the corresponding path from the final precedence matrix $[P^n]$.

Steps of Floyd's algorithm

Step 1: Set $k = 0$.

Step 2: Form the initial distance matrix $[D^0]$ and the initial precedence matrix $[P^0]$ from the distance network.

The distance matrix as well as the precedence matrix are with n rows and n columns. In the distance matrix, between any two nodes, if there is no direct arc, then the corresponding distance is assumed as ∞ . The diagonal values of the initial distance matrix are assumed as 0. In the initial precedence matrix, the diagonal values are assumed as '-' and the other entries in i th row are assumed as i , for $i = 1, 2, 3, \dots, n$.

Step 3: Set $k = k + 1$.

Step 4: Obtain the values of the distance matrix, $[D^k]$ for all the cells where i is not equal to j using the following formula.

$$D_{ij}^k = \min \left[D_{ij}^{k-1}, \left(D_{ik}^{k-1} + D_{kj}^{k-1} \right) \right]$$

Step 5: Obtain the values of the precedence matrix, $[P^k]$, for all the cells where i is not equal to j using the following formula.

$$P_{ij}^k = \begin{cases} P_{ij}^{k-1}, & \text{if } D_{ij}^k \text{ is not equal to } D_{ij}^{k-1} \\ P_{ij}^{k-1}, & \text{otherwise.} \end{cases}$$

Step 6: If $k = n$, go to Step 7; otherwise, set $k = k + 1$ and go to Step 4.

Step 7: For each source-destination nodes combination, as required in reality, find the shortest distance from the final distance matrix, $[D^n]$ and trace the corresponding shortest path from the final precedence matrix $[P^n]$. Guidelines for tracing the shortest path for a given combination of source node and destination node are presented below.

7.1 Let the source node be X and the destination node be Y .

7.2 Fix Node Y as the last node in the partially formed shortest path.

7.3 Find the value from the final precedence matrix $[P^n]$ for the row corresponding to Node X and the column corresponding to Y . Let it be Q . Prefix Node Q in the partially formed shortest path.

7.4 Check whether Q is equal to X . If not so, set $Y = Q$ and go to Step 7.3; otherwise go to Step 7.5.

7.5 The path constructed is the required shortest path from the source Node X to the destination Node Y .

Example 5.3 Consider the distance network as shown in Fig. 5.20.

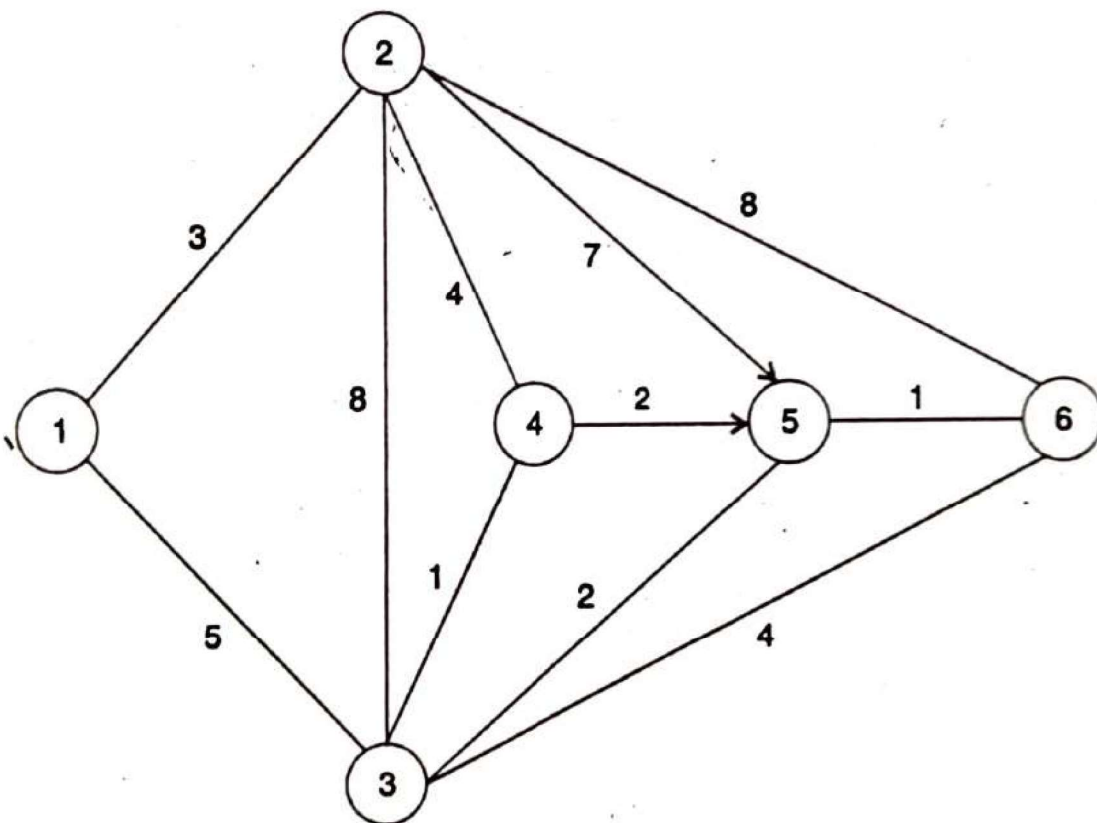


Fig. 5.20 Distance network.

- Apply Floyd's algorithm to it and generate the final distance matrix and precedence matrix.
- Find the shortest path and the corresponding distance from the source node to the destination node as indicated in each of the cases: 1-6, 5-1 and 5-2.

Solution Applying Floyd's algorithm, we get Tables 5.17-5.23.

Iteration 0

Table 5.17(a) Distance Matrix [D^0]

	1	2	3	4	5	6
1	0	3	5	∞	∞	∞
2	3	0	8	4	7	8
3	5	8	0	1	2	4
4	∞	4	1	0	2	∞
5	∞	∞	2	∞	0	1
6	∞	8	4	∞	1	0

Table 5.17(b) Precedence Matrix [P^0]

	1	2	3	4	5	6
1	-	1	1	1	1	1
2	2	-	2	2	2	2
3	3	3	-	3	3	3
4	4	4	4	-	4	4
5	5	5	5	5	-	5
6	6	6	6	6	6	-

Iteration 1

Table 5.18(a) Distance Matrix [D^1]

	1	2	3	4	5	6
1	0	3	5	∞	∞	∞
2	3	0	8	4	7	8
3	5	8	0	1	2	4
4	∞	4	1	0	2	∞
5	∞	∞	2	∞	0	1
6	∞	8	4	∞	1	0

Table 5.18(b) Precedence Matrix [P^1]

	1	2	3	4	5	6
1	-	1	1	1	1	1
2	2	-	2	2	2	2
3	3	3	-	3	3	3
4	4	4	4	-	4	4
5	5	5	5	5	-	5
6	6	6	6	6	6	-

Iteration 2

Table 5.19(a) Distance Matrix [D^2]

	1	2	3	4	5	6
1	0	3	5	7	10	11
2	3	0	8	4	7	8
3	5	8	0	1	2	4
4	7	4	1	0	2	12
5	∞	∞	2	∞	0	1
6	11	8	4	12	1	0

Table 5.19(b) Precedence Matrix [P^2]

	1	2	3	4	5	6
1	-	1	1	2	2	2
2	2	-	2	2	2	2
3	3	3	-	3	3	3
4	2	4	4	-	4	2
5	5	5	5	5	-	5
6	2	6	6	2	6	-

Iteration 3

Table 5.20(a) Distance Matrix [D^3]

	1	2	3	4	5	6
1	0	3	5	6	7	9
2	3	0	8	4	7	8
3	5	8	0	1	2	4
4	6	4	1	0	2	5
5	7	10	2	3	0	1
6	9	8	4	5	1	0

Table 5.20(b) Precedence Matrix [P^3]

	1	2	3	4	5	6
1	-	1	1	3	3	3
2	2	-	2	2	2	2
3	3	3	-	3	3	3
4	3	4	4	-	4	3
5	3	3	5	3	-	5
6	3	6	6	3	6	-

Iteration 4

Table 5.21(a) Distance Matrix [D^4]

	1	2	3	4	5	6
1	0	3	5	6	7	9
2	3	0	5	4	6	8
3	5	5	0	1	2	4
4	6	4	1	0	2	5
5	7	7	2	3	0	1
6	9	8	4	5	1	0

Table 5.21(b) Precedence Matrix [P^4]

	1	2	3	4	5	6
1	-	1	1	3	3	3
2	2	-	4	2	4	2
3	3	4	-	3	3	3
4	3	4	4	-	4	3
5	3	4	5	3	-	5
6	3	6	6	3	6	-

Iteration 5

Table 5.22(a) Distance Matrix [D^5]

	1	2	3	4	5	6
1	0	3	5	6	7	8
2	3	0	5	4	6	7
3	5	5	0	1	2	3
4	6	4	1	0	2	3
5	7	7	2	3	0	1
6	8	8	3	4	1	0

Table 5.22(b) Precedence Matrix [P^5]

	1	2	3	4	5	6
1	-	1	1	3	3	5
2	2	-	4	2	4	5
3	3	4	-	3	3	5
4	3	4	4	-	4	5
5	3	4	5	3	-	5
6	3	6	5	3	6	-

Iteration 6

Table 5.23(a) Distance Matrix [D^6]

	1	2	3	4	5	6
1	0	3	5	6	7	8
2	3	0	5	4	6	7
3	5	5	0	1	2	3
4	6	4	1	0	2	3
5	7	7	2	3	0	1
6	8	8	3	4	1	0

Table 5.23(b) Precedence Matrix [P^6]

	1	2	3	4	5	6
1	-	1	1	3	3	5
2	2	-	4	2	4	5
3	3	4	-	3	3	5
4	3	4	4	-	4	5
5	3	4	5	3	-	5
6	3	6	5	3	6	-

(b) The shortest path from Node 1 to Node 6 is 1-3-5-6; the corresponding distance = 8. The shortest path from Node 5 to Node 1 is 5-3-1; the corresponding distance = 7. The shortest path from Node 5 to Node 2 is 5-3-4-2; the corresponding distance = 7.

5.3 MINIMUM SPANNING TREE PROBLEM

Let us consider the example of laying telephone cable in a locality as shown in Fig. 5.21 which summarizes the distance network of the locality. The number on each arc represents the distance between the nodes connected by that arc.

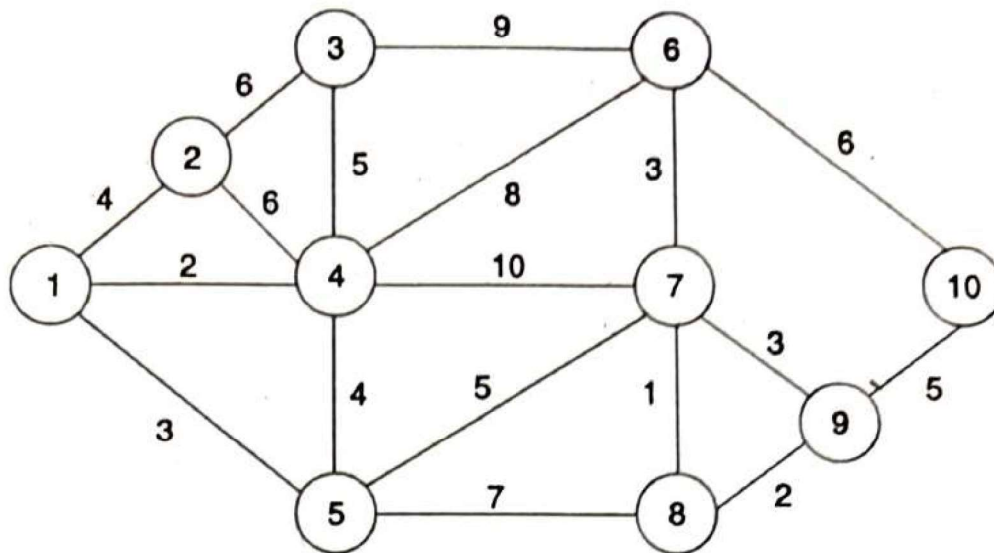


Fig. 5.21 Distance network.

The objective of the minimum spanning tree problem is to connect the nodes of the network by a set of arcs such that the total length of all the arcs is minimized. Therefore, in the case of laying the telephone cable, the objective is to connect all the nodes by a set of arcs such that the total length of telephone cable to be laid is minimized.

In this section, the two algorithms for the minimum spanning tree problem are presented: *PRIM algorithm* and *Kruskal's algorithm*. These are discussed in the next sections.

5.3.1 PRIM Algorithm

PRIM algorithm is used to find solution for the minimum spanning tree problem. The steps of this algorithm are presented below.

- Step 1:* Represent the distance network (Fig. 5.21) in a matrix form (Table 5.24).
- Step 2:* Let Q is a null set, which is the set of selected row numbers of the matrix.
- Step 3:* Select Row 1 and include it in Q . Then, delete Column 1 of the matrix.
- Step 4:* Find the minimum of the undeleted values among the rows in Q and select it by marking a square around it. (In case of tie, break it randomly.)
- Step 5:* Identify the corresponding column number (K) and then include Row K in Q .
- Step 6:* Delete Column K of the matrix.
- Step 7:* Check whether all the columns are deleted. If yes, go to Step 8; otherwise go to Step 4.
- Step 8:* Show the arcs in the spanning tree corresponding to the cells of the matrix marked with squares by thick lines.
- Step 9:* Find the total of all the values (marked with squares). This is the minimized total length of the arcs to connect all the nodes of the network as per the minimum spanning tree concept.

Example 5.4 Consider the distance network shown in Fig. 5.21. Find the minimum spanning tree of this network using the PRIM algorithm.

Solution

Step 1: The distance network is shown in Table 5.24.

Table 5.24 Example 5.4

	1	2	3	4	5	6	7	8	9	10
1	—	4	∞	2	3	∞	∞	∞	∞	∞
2	4	—	6	6	∞	∞	∞	∞	∞	∞
3	∞	6	—	5	∞	9	∞	∞	∞	∞
4	2	6	5	—	4	8	10	∞	∞	∞
5	3	∞	∞	4	—	∞	5	7	∞	∞
6	∞	∞	9	8	∞	—	3	∞	∞	6
7	∞	∞	∞	10	5	3	—	1	3	∞
8	∞	∞	∞	∞	7	∞	1	—	2	∞
9	∞	∞	∞	∞	∞	∞	3	2	—	5
10	∞	∞	∞	∞	∞	6	∞	∞	5	—

Step 2: Let Q is a null set.

Step 3: $Q = \{1\}$. The matrix after deleting Column 1 is shown as in Table 5.25, where each of the rows in Q is indicated by '*'.

Table 5.25 Distance Matrix After Deleting Column 1

	2	3	4	5	6	7	8	9	10
*1	4	∞	2	3	∞	∞	∞	∞	∞
2	—	6	6	∞	∞	∞	∞	∞	∞
3	6	—	5	∞	9	∞	∞	∞	∞
4	6	5	—	4	8	10	∞	∞	∞
5	∞	∞	4	—	∞	5	7	∞	∞
6	∞	9	8	∞	—	3	∞	∞	6
7	∞	∞	10	5	3	—	1	3	∞
8	∞	∞	∞	7	∞	1	—	2	∞
9	∞	∞	∞	∞	∞	3	2	—	5
10	∞	∞	∞	∞	6	∞	∞	5	—

Step 4: Minimum of the undeleted cell values of the rows in Q is 2. So, it is marked with a square as shown in Table 5.25.

Step 5: $K = 4$ and $Q = \{1, 4\}$.

Step 6: The matrix after deleting Column 4 is shown in Table 5.26.

Table 5.26 Distance Matrix After Deleting Column 4

	2	3	5	6	7	8	9	10
*1	4	∞	3	∞	∞	∞	∞	∞
2	—	6	∞	∞	∞	∞	∞	∞
3	6	—	∞	9	∞	∞	∞	∞
*4	6	5	4	8	10	∞	∞	∞
5	∞	∞	—	∞	5	7	∞	∞
6	∞	9	∞	—	3	∞	∞	6
7	∞	∞	5	3	—	1	3	∞
8	∞	∞	7	∞	1	—	2	∞
9	∞	∞	∞	∞	3	2	—	5
10	∞	∞	∞	6	∞	∞	5	—

Step 7: All the columns are not deleted. So, go to Step 4.

Step 4: Minimum of the undeleted cell values of the rows in Q is 3. So, it is marked with a square as shown in Table 5.26.

Step 5: $K = 5$ and $Q = \{1, 4, 5\}$.

Step 6: The matrix after deleting Column 5 is shown as in Table 5.27.

Table 5.27 Distance Matrix After Deleting Column 5

	2	3	6	7	8	9	10
*1	4	∞	∞	∞	∞	∞	∞
2	—	6	∞	∞	∞	∞	∞
3	6	—	9	∞	∞	∞	∞
*4	6	5	8	10	∞	∞	∞
*5	∞	∞	∞	5	7	∞	∞
6	∞	9	—	3	∞	∞	6
7	∞	∞	3	—	1	3	∞
8	∞	∞	∞	1	—	2	∞
9	∞	∞	∞	3	2	—	5
10	∞	∞	6	∞	∞	5	—

Step 7: All the columns are not deleted. So, go to Step 4.

Step 4: Minimum of the undeleted cell values of the rows in Q is 4. So, it is marked with a square as shown in Table 5.27.

Step 5: $K = 2$ and $Q = \{1, 4, 5, 2\}$.

Step 6: The matrix after deleting Column 2 is shown as in Table 5.28.

Table 5.28 Distance Matrix After Deleting Column 2

	3	6	7	8	9	10
*1	∞	∞	∞	∞	∞	∞
*2	6	∞	∞	∞	∞	∞
3	—	9	∞	∞	∞	∞
*4	5	8	10	∞	∞	∞
*5	∞	∞	5	7	∞	∞
6	9	—	3	∞	∞	6
7	∞	3	—	1	3	∞
8	∞	∞	1	—	2	∞
9	∞	∞	3	2	—	5
10	∞	6	∞	∞	5	—

Step 7: All the columns are not deleted. So, go to Step 4.

Step 4: Minimum of the undeleted cell values of the rows in Q is 5. (This minimum is for Column 3 and Row 4 as well as Column 7 and Row 5. One of these two locations is selected randomly.) So, it is marked with a square at the cell with respect to the Row 5 and the Column 7 as shown in Table 5.28.

Step 5: $K = 7$ (random selection) and $Q = \{1, 4, 5, 2, 7\}$.

Step 6: The matrix after deleting Column 7 is shown as in Table 5.29.

Table 5.29 Distance Matrix After Deleting Column 7

	3	6	8	9	10
*1	∞	∞	∞	∞	∞
*2	6	∞	∞	∞	∞
3	—	9	∞	∞	∞
*4	5	8	∞	∞	∞
*5	∞	∞	7	∞	∞
6	9	—	∞	∞	6
*7	∞	3	1	3	∞
8	∞	∞	—	2	∞
9	∞	∞	2	—	5
10	∞	6	∞	5	—

Step 7: All the columns are not deleted. So, go to Step 4.

Step 4: Minimum of the undeleted cell values of the rows in Q is 1. So, it is marked with a square as shown in Table 5.29.

Step 5: $K = 8$ and $Q = \{1, 4, 5, 2, 7, 8\}$.

Step 6: The matrix after deleting Column 8 is shown as in Table 5.30.

Table 5.30 Distance Matrix After Deleting Column 8

	3	6	9	10
*1	∞	∞	∞	∞
*2	6	∞	∞	∞
3	—	9	∞	∞
*4	5	8	∞	∞
*5	∞	∞	∞	∞
6	9	—	∞	6
*7	∞	3	3	∞
*8	∞	∞	2	∞
9	∞	∞	—	5
10	∞	6	5	—

Step 7: All the columns are not deleted. So, go to Step 4.

Step 4: Minimum of the undeleted values of the rows in Q is 2. So, it is marked with a square as shown in Table 5.30.

Step 5: $K = 9$ and $Q = \{1, 4, 5, 2, 7, 8, 9\}$.

Step 6: The matrix after deleting Column 9 is shown as in Table 5.31.

Table 5.31 Distance Matrix After Deleting Column 9

	3	6	10
*1	∞	∞	∞
*2	6	∞	∞
3	—	9	∞
*4	5	8	∞
*5	∞	∞	∞
6	9	—	6
*7	∞	3	∞
*8	∞	∞	∞
*9	∞	∞	5
10	∞	6	—

Step 7: All the columns are not deleted. So, go to Step 4.

Step 4: Minimum of the undeleted cell values of the rows in Q is 3. So, it is marked with a square as shown in Table 5.31.

Step 5: $K = 6$ and $Q = \{1, 4, 5, 2, 7, 8, 9, 6\}$.

Step 6: The matrix after deleting Column 6 is shown as in Table 5.32.

Table 5.32 Distance Matrix After Deleting Column 6

	3	10
*1	∞	∞
*2	6	∞
3	—	∞
*4	5	∞
*5	∞	∞
*6	9	6
*7	∞	∞
*8	∞	∞
*9	∞	5
10	∞	—

Step 7: All the columns are not deleted. So, go to Step 4.

Step 4: Minimum of the undeleted cell values of the rows in Q is 5, for Column 3 and Row 4 as well as Column 10 and row 9. So, it is marked with a square as shown in Table 5.32.

Step 5: $K = 10$ (randomly selected) $Q = \{1, 4, 5, 2, 7, 8, 9, 6, 10\}$.

Step 6: The matrix after deleting Column 10 is shown as in Table 5.33.

Table 5.33 Distance Matrix After Deleting Column 10

	3
*1	∞
*2	6
3	—
*4	5
*5	∞
*6	9
*7	∞
*8	∞
*9	∞
*10	∞

Step 7: All the columns are not deleted. So, go to Step 4.

Step 4: Minimum of the undeleted cell values of the rows in Q is 5. So, it is marked with a square as shown in Table 5.33.

Step 5: $K = 3$ and $Q = \{1, 4, 5, 2, 7, 8, 9, 6, 10, 3\}$.

Step 6: After deleting Column 3, the matrix is completely deleted.

Step 7: All the columns are deleted. So, go to Step 8.

Step 8: The minimum spanning tree is shown in Fig. 5.22 and the corresponding minimum total length of the arcs is 30.

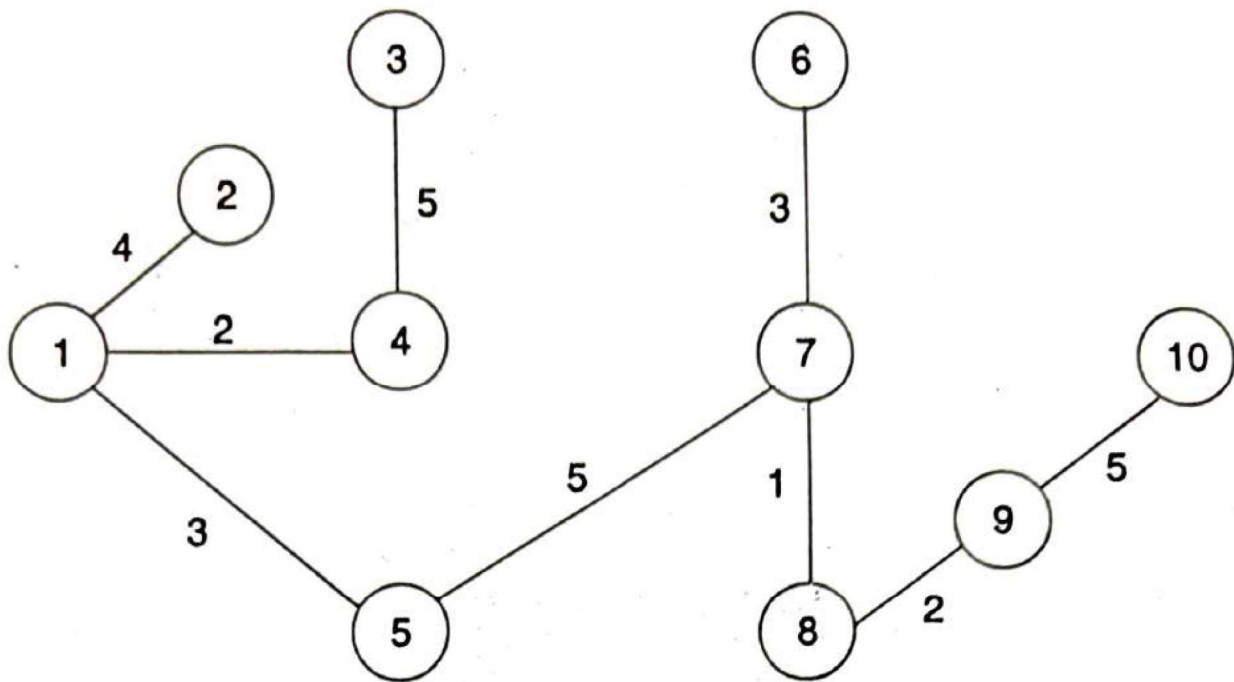


Fig. 5.22 Minimum spanning tree.

5.3.2 Kruskal's Algorithm

Consider the arcs of the distance network of the given problem. If the network is undirected, then it is sufficient to form a table summarizing the distance of each arc (i, j) in the network only for $i < j$.

- Step 1:** Arrange the arcs along with their distances as per ascending order of their distances. Let this arrangement be called as set M .
- Step 2:** Create a graph with n nodes without arcs.
- Step 3:** Select the first undeleted arc from the set M .
- Step 4:** Check whether the inclusion of the selected arc in the previous step into the graph forms a cycle. If yes, go to Step 6; otherwise, go to Step 5.
- Step 5:** Include the selected arc in the graph.
- Step 6:** Delete that arc from the set M .
- Step 7:** Check whether all the nodes in the graph are connected. If no, go to Step 3; otherwise go to Step 8.
- Step 8:** Stop.

Example 5.5 Consider the network given as in Fig. 5.23. Find the minimum spanning tree using Kruskal's algorithm.

Solution Since, the network is undirected, the distance of each arc (i, j) , for $i < j$ is presented as in Table 5.34.

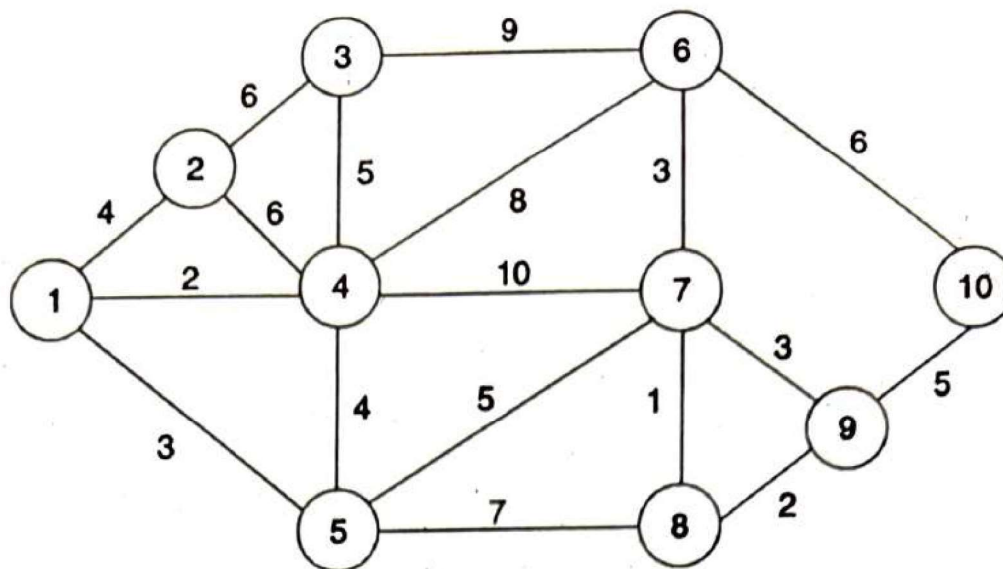


Fig. 5.23 Distance network.

Table 5.34 Distances of the Arcs

Arc	Distance	Arc	Distance
1-2	4	4-7	10
1-4	2	5-7	5
1-5	3	5-8	7
2-3	6	6-7	3
2-4	6	6-10	6
3-4	5	7-8	1
3-6	9	7-9	3
4-5	4	8-9	2
4-6	8	9-10	5

Step 1. Sort the distances of the arcs (refer to Table 5.35).

Table 5.35 Sorted Distances of Arcs

Arc	Distance	Arc	Distance
7-8	1	5-7	5
1-4	2	9-10	5
8-9	2	2-3	6
1-5	3	2-4	6
6-7	3	6-10	6
7-9	3	5-8	7
1-2	4	4-6	8
4-5	4	3-6	9
3-4	5	4-7	10

Steps 2 onwards: Figures 5.24 through 5.32 show the Steps 2 onwards. The arcs from 7-8 up to 9-10 except the arcs 7-9 and 4-5 in Table 5.35 are included in the spanning tree which is shown as in Fig. 5.32 to connect all the nodes of the problem. Inclusion of the arcs 7-9 and 4-5 creates loops. Hence, they are not included. The sum of the distances of the arcs in the spanning tree which is shown as in Fig. 5.32 is 30.

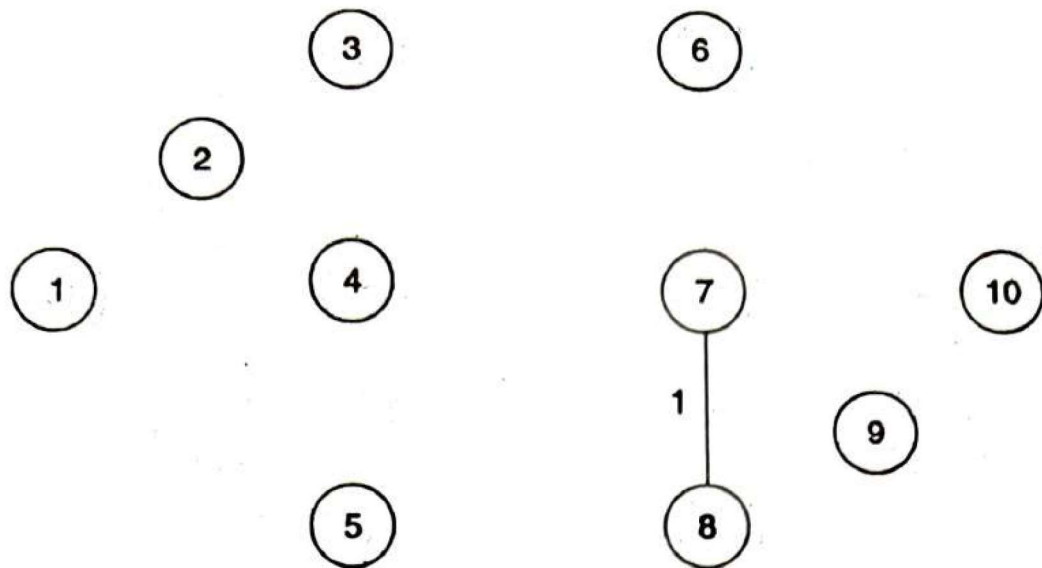


Fig. 5.24 Partial spanning tree after adding arc 7-8.

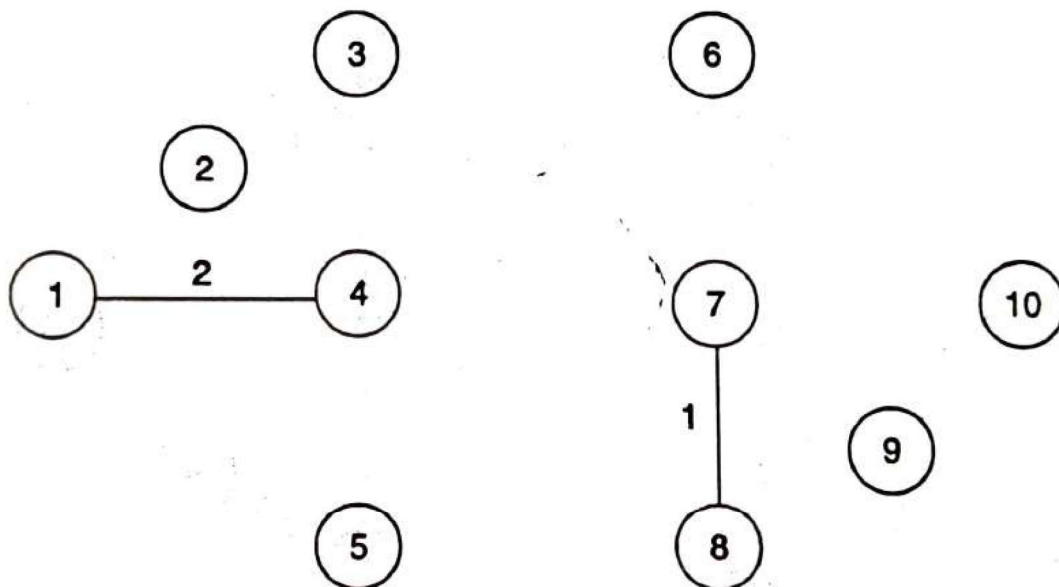


Fig. 5.25 Partial spanning tree after adding arc 1-4.

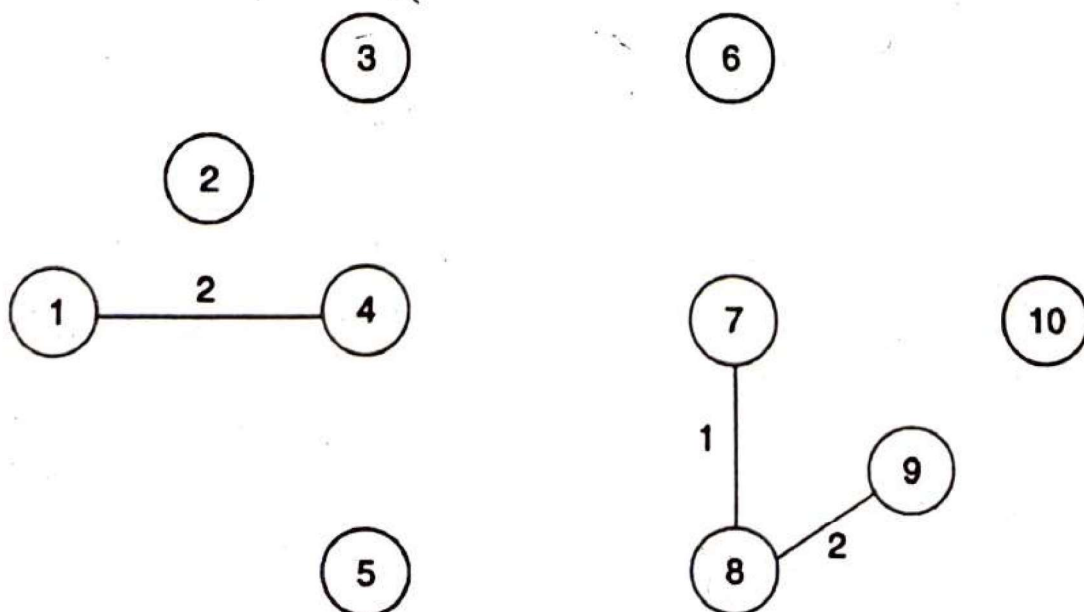


Fig. 5.26 Partial spanning tree after adding arc 8-9.

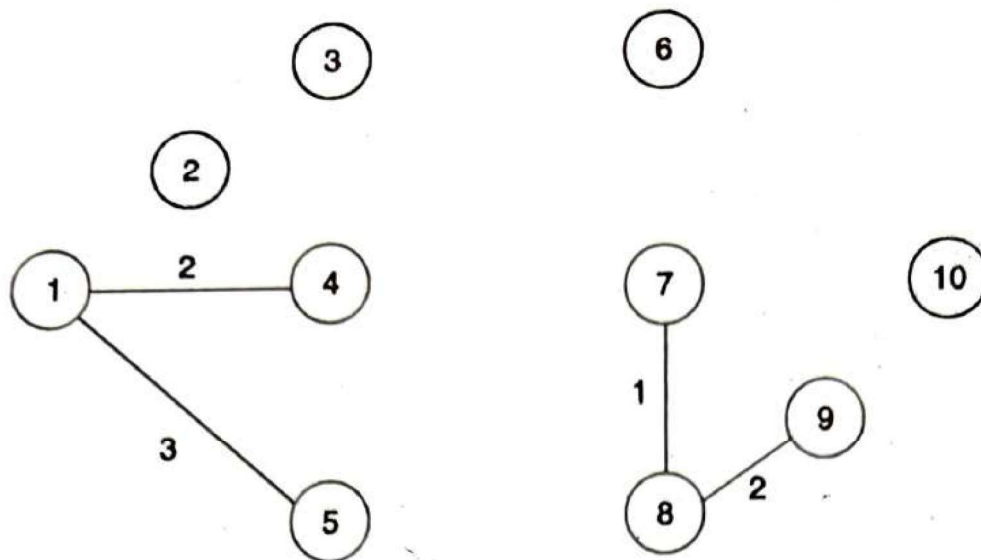


Fig. 5.27 Partial spanning tree after adding arc 1-5.

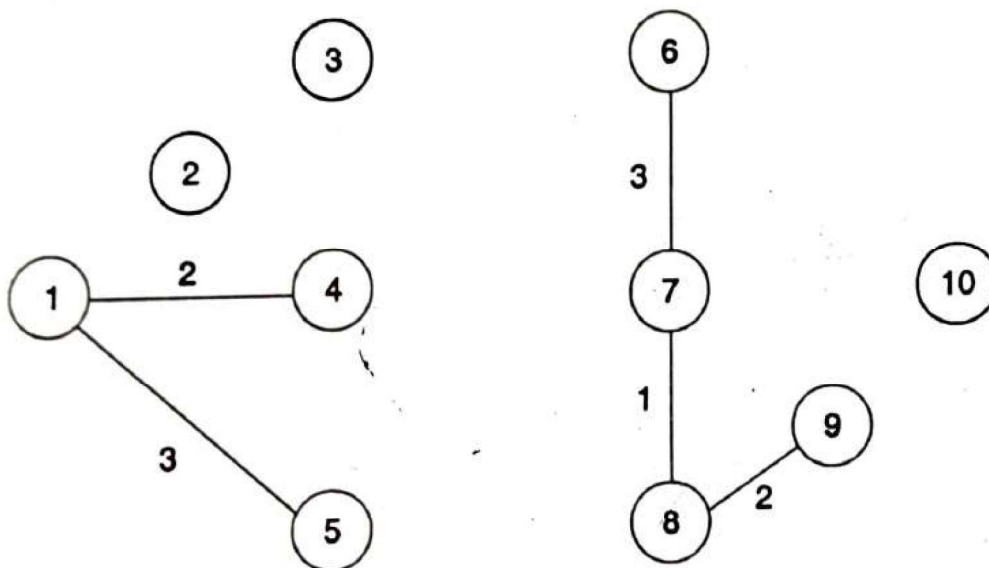


Fig. 5.28 Partial spanning tree after adding arc 6-7.

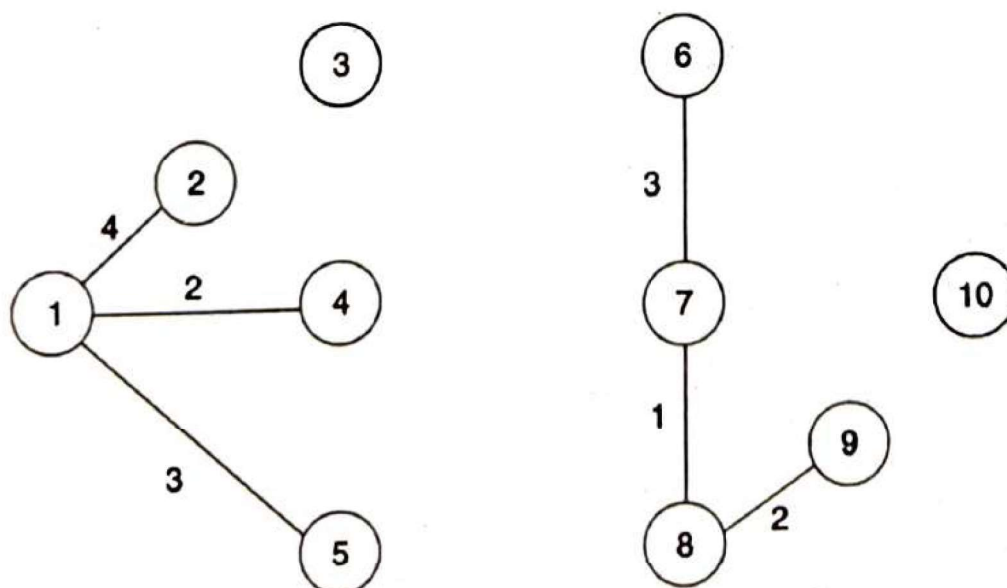


Fig. 5.29 Partial spanning tree after adding arc 1-2.

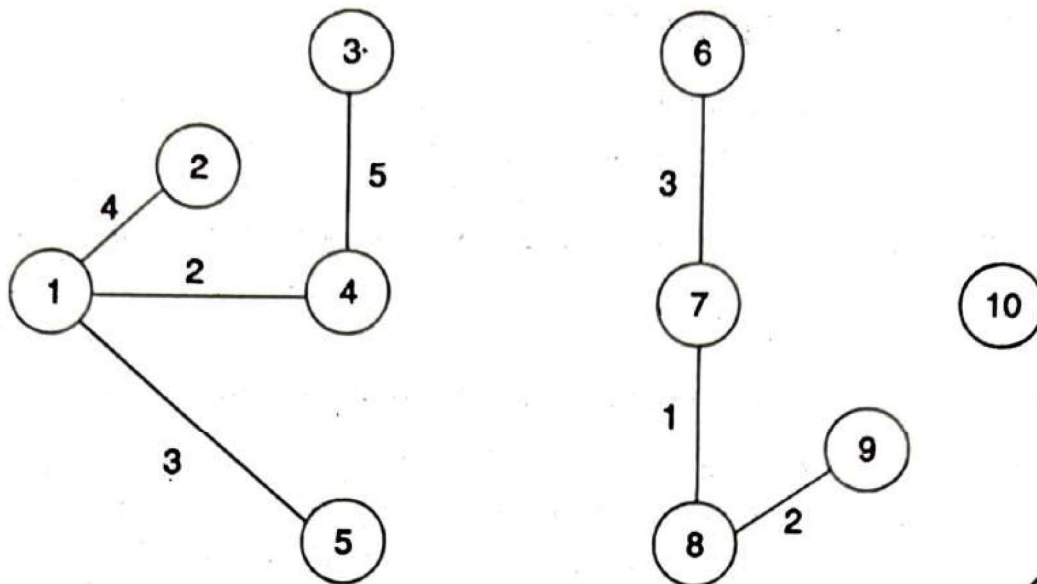


Fig. 5.30 Partial spanning tree after adding arc 3-4.

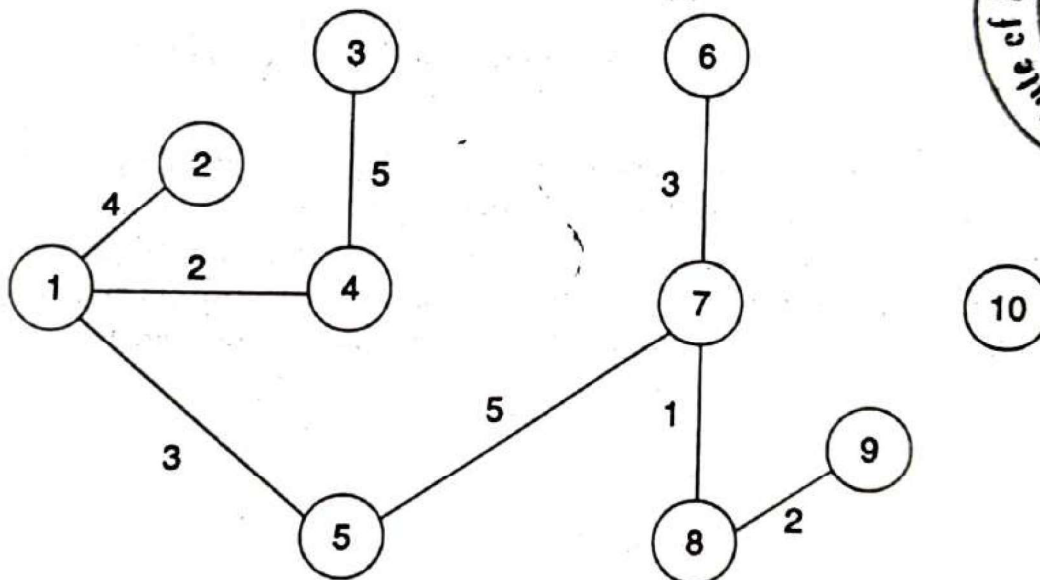


Fig. 5.31 Partial spanning tree after adding arc 5-7.

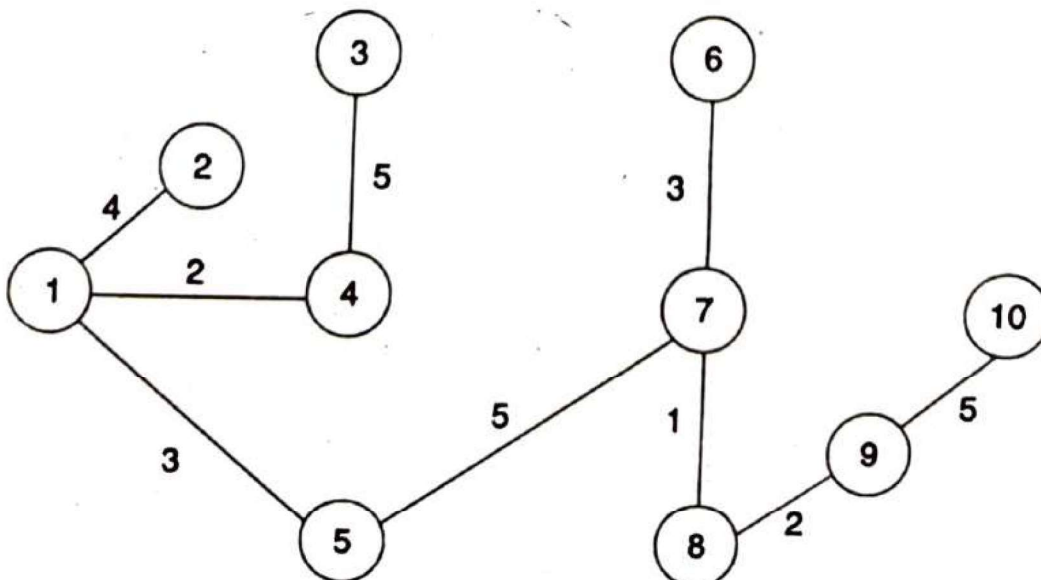


Fig. 5.32 Minimum spanning tree after adding arc 9-10 (Final).



5.4 MAXIMAL FLOW PROBLEM

In this section, we consider the situation of a pipe network used to transfer fluid (oil, water, etc.) from one location to another location. The maximum flow of fluid in each pipe segment will be limited to a particular value depending on the diameter of the pipe in that segment or the slope of the pipe in the segment. The arc $i-j$ (pipe segment between any two locations i and j) of the network will have the following data:

1. Maximum permitted flow of fluid per unit time from the Node i to the Node j .
2. Maximum permitted flow of fluid per unit time from the Node j to the Node i .

A meaningful objective of this problem is to determine the maximum flow of fluid from a given source node to a given destination node. The maximal flow problem can be solved by using two methods: linear programming modelling and maximal flow algorithm. These are discussed in the following sections.

5.4.1 Linear Programming Modelling of Maximal Flow Problem

The modelling of the maximal flow problem as a linear programming model is demonstrated with the following example:

Example 5.6 Consider the flow network shown as in Fig. 5.33 which gives information about capacities of flow of oil in both ways between different pairs of locations. For example, in the arc 1-2, the number 5 nearer to Node 1 represents the maximal flow permitted per unit time from Node 1 to Node 2. Similarly, the number 0 marked nearer to Node 2 represents the maximal flow permitted from Node 2 to Node 1. Find the maximum flow of oil from the location 1 to the location 5 using a linear programming model.

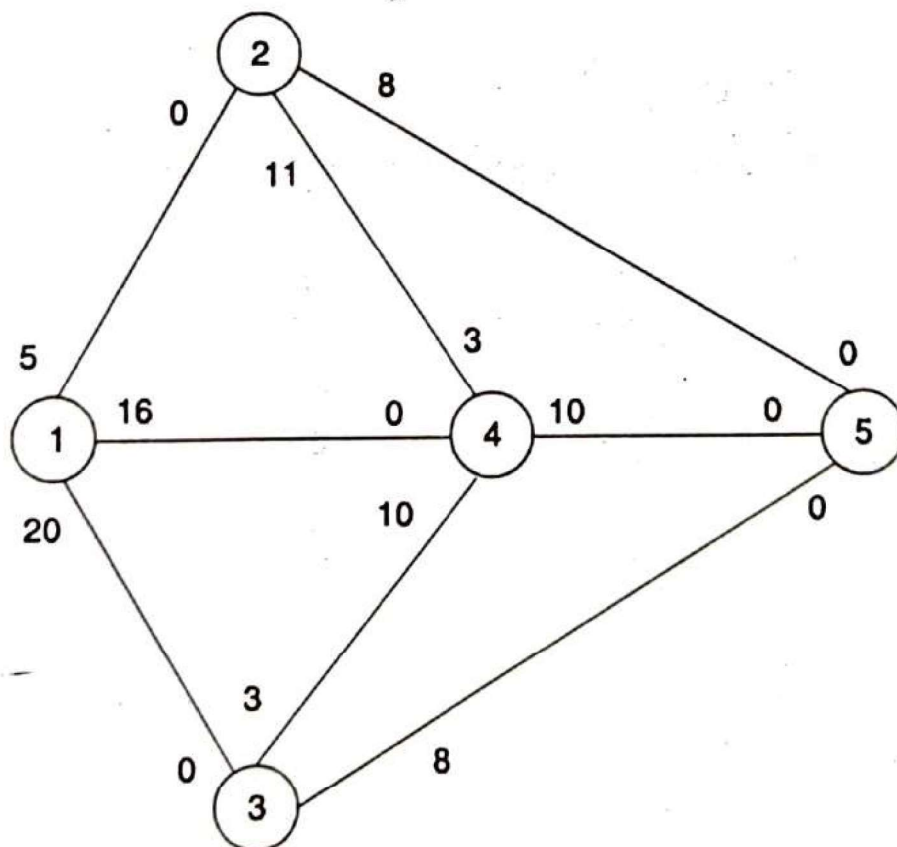


Fig. 5.33 Network with flow capacities.

Solution Let us assume that the flow admitted into Node 1 is equal to f . So, the flow out of Node 5 is f . The different variables representing flows (forward as well as backward) on each of the arcs are summarized in Fig. 5.34. In an arc (i, j) , the flow capacity from the Node i to Node j is represented nearer to the Node i and the flow capacity from the Node j to Node i is represented nearer to Node j . For every Node, the total quantity of flow into that node is equal to the total quantity of flow out of that node. On a given arc, if the flow in a particular direction is 0, there is no need to assign a variable to represent the flow in that direction.

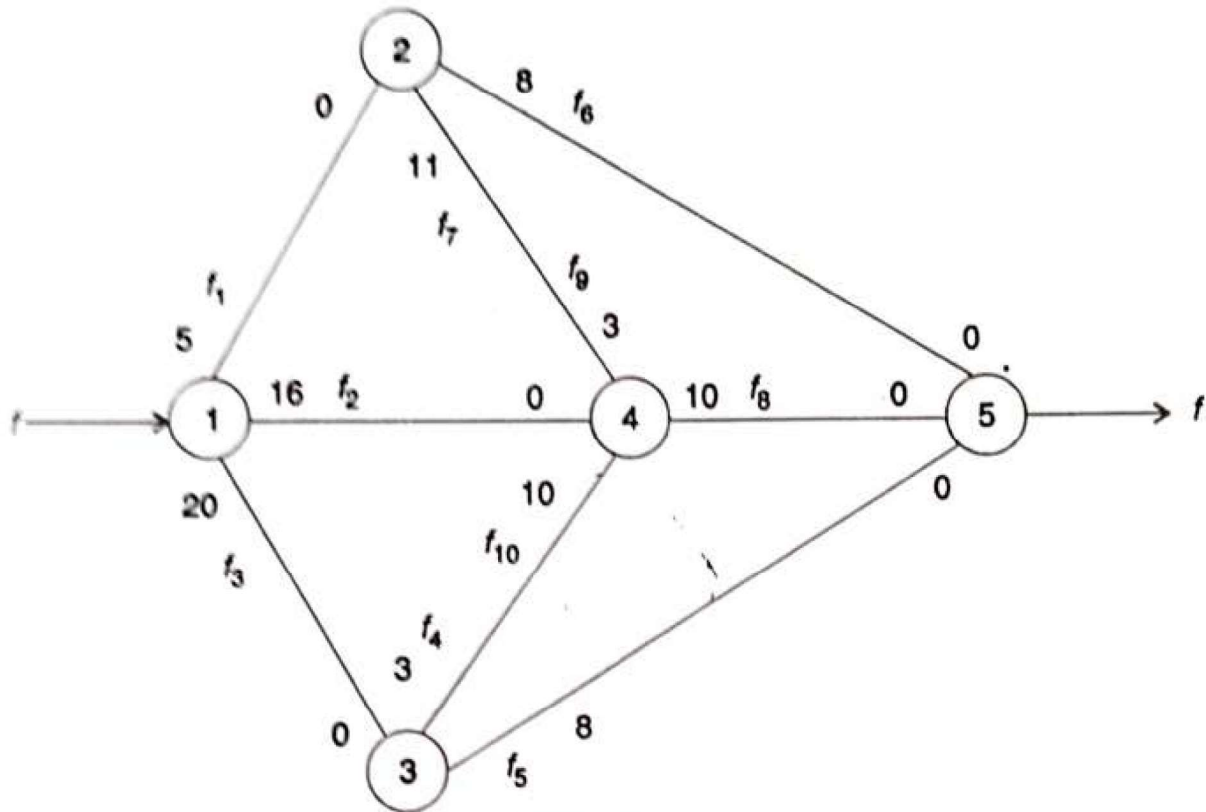


Fig. 5.34 Network with flow capacities.

A linear programming model for the above problem to maximize the flow from Node 1 to Node 5 is presented below.

$$\text{Maximize } Z = f$$

subject to

$$\begin{aligned} f &= f_1 + f_2 + f_3 && \text{(at Node 1)} \\ f_1 + f_9 &= f_7 + f_6 && \text{(at Node 2)} \\ f_3 + f_{10} &= f_4 + f_5 && \text{(at Node 3)} \\ f_2 + f_7 + f_4 &= f_8 + f_9 + f_{10} && \text{(at Node 4)} \\ f_6 + f_8 + f_5 &= f && \text{(at Node 5)} \\ f_1 &\leq 5 \\ f_2 &\leq 16 \\ f_3 &\leq 20 \\ f_4 &\leq 3 \\ f_5 &\leq 8 \end{aligned}$$

$$\begin{aligned}
 f_6 &\leq 8 \\
 f_7 &\leq 11 \\
 f_8 &\leq 10 \\
 f_9 &\leq 3 \\
 f_{10} &\leq 10
 \end{aligned}$$

where

$$f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9 \text{ and } f_{10} \geq 0$$

The results of the above model are as under:

$$f = 26, f_1 = 5, f_2 = 13, f_3 = 8, f_5 = 8, f_6 = 8, f_8 = 10 \text{ and } f_9 = 3$$

Hence, the maximum flow of oil from Node 1 to Node 5 is 26 units. The results are shown as in Fig. 5.35.

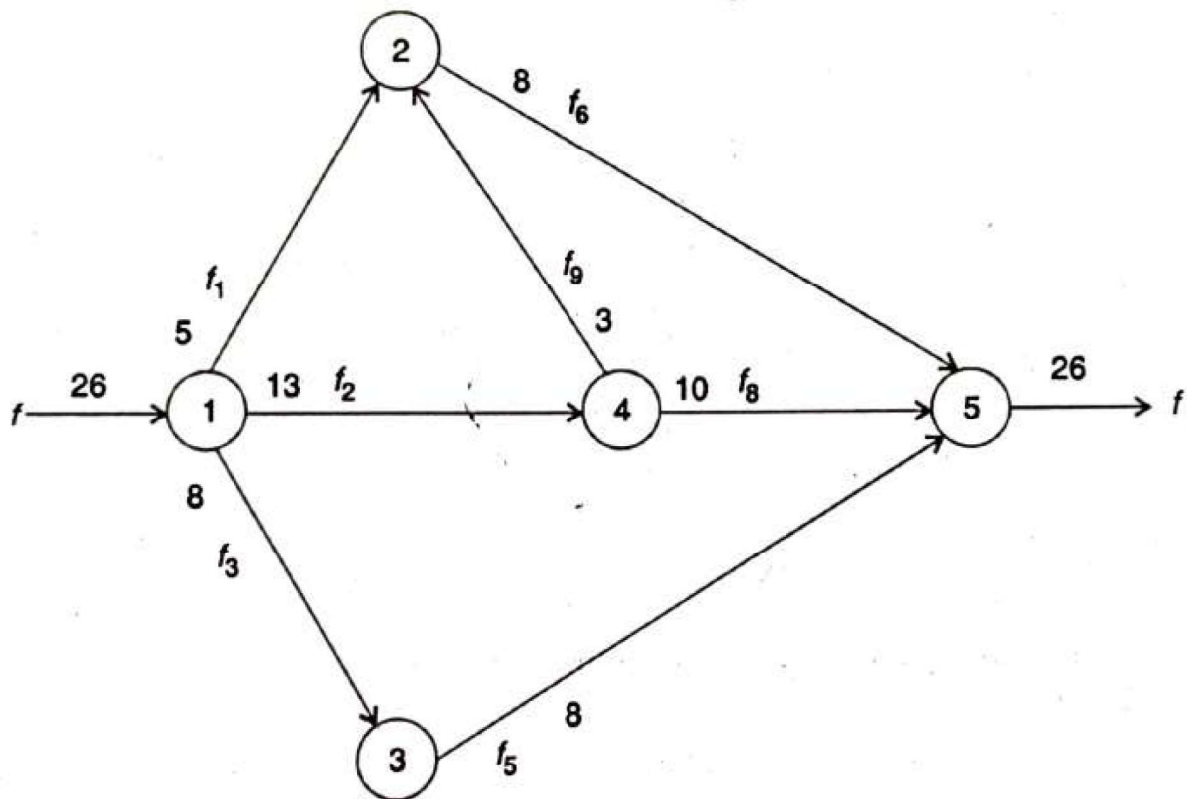


Fig. 5.35 Results as per maximum flow from Node 1 to Node 5.

5.4.2 Maximal Flow Problem (MFP) Algorithm

The objective of this problem is to find the maximal flow from a given source (r) to a given destination (s). The algorithm to solve such problems is presented as follows:

- Step 1:** Form a FROM-TO capacity matrix $[A]$ for the given flow network. A_{ij} is the flow from the Node i to the Node j ($i = 1, 2, 3, \dots, n$ and $j = 1, 2, 3, \dots, n$, where n is the total number of nodes in the network). Set the iteration number, $k = 1$, Cumulative flow, $X = 0$).
- Step 2:** Trace a path starting from the given source (r) to the given destination (s) directly or passing through some intermediate nodes which will have some feasible quantity of flow.
- Step 3:** Check whether such path exists. If yes, go to Step 4; otherwise, go to Step 7.

Step 4: Find the minimum of the capacities/remaining capacities of various links of the path traced in Step 2. Let this quantity be Q_k .

$$\text{Set, } X = X + Q_k$$

(Note: While tracing a path in Step 2, enough care should be taken to maximize Q_k)

Step 5: Obtain the next flow matrix by performing the following steps.

- Subtract Q_k from all A_{ij} values corresponding to all forward links of the path traced.
- Add Q_k to all the A_{ij} values corresponding to all backward links of the path traced.

Step 6: Set $k = k + 1$ and go to Step 2.

Step 7: Obtain a new flow matrix $[C]$ by subtracting the elements of the flow matrix $[B]$ in the last iteration from the corresponding elements of the flow matrix $[A]$ in the first iteration.

$$C_{ij} = A_{ij} - B_{ij}, \text{ if } A_{ij} > B_{ij} \quad \text{for all values of } i \text{ and } j;$$

$$= 0, \text{ otherwise.}$$

Step 8: (a) The cell entries of the flow matrix $[C]$ represent the flows in various arcs of the network.

(b) The maximal flow from the source node (r) to the destination node (s) is X .

(c) Map the cell entries on to the corresponding arcs of the network showing the actual flows in various arcs to achieve the maximal flow of X .

Example 5.7 Consider the pipe network shown as in Fig. 5.36 showing the flow capacities between various pairs of locations in both ways. Find the maximal flow from Node 1 to Node 6.

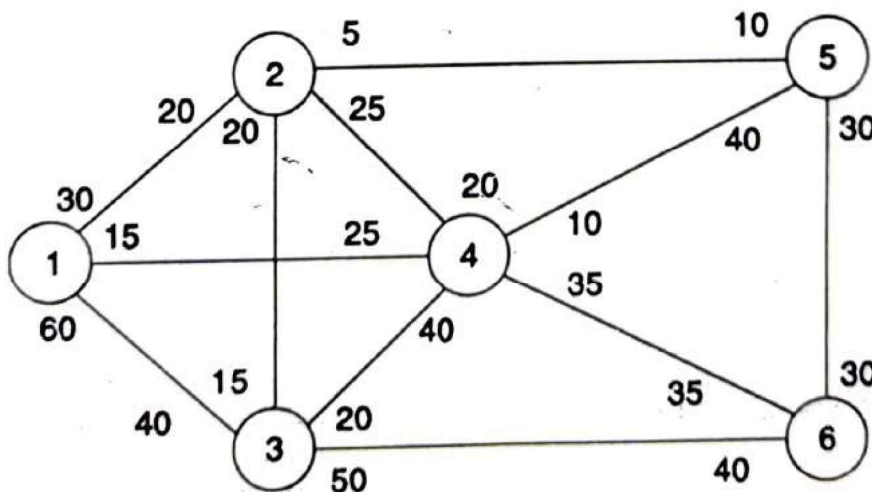


Fig. 5.36 Network showing pipe capacities.

Solution

Iteration 1. The data of the Fig. 5.36 is shown in matrix form as in Table 5.36.

Table 5.36 Flow Matrix of Iteration 1

		To					
		1	2	3	4	5	6
From	1	—	30	60 [—]	15	—	—
	2	20	—	20	25	5	—
	3	40 ⁺	15	—	20	—	50 [—]
	4	25	20	40	—	10	35
	5	—	10	—	40	—	30
	6	—	—	40 ⁺	35	30	—

To start with, starting from Node 1, the following path is assumed.

$$\text{Path 1} = 1-3-6 \quad \text{and} \quad Q_1 = \min(60 \text{ and } 50) = 50$$

Therefore, the feasible quantity of flow through the path is 50. i.e. $X = 50$.

Iteration 2. Now, assign minus sign to the cells 1-3 and 3-6 and then assign positive sign to the cells 3-1 and 6-3. Obtain Table 5.37 of Iteration 2 by subtracting Q_1 from all the negatively-signed cells and adding Q_1 to all the positively-signed cells of Table 5.36.

Table 5.37 Flow Matrix of Iteration 2

		To					
		1	2	3	4	5	6
From	1	—	30 [—]	10	15	—	—
	2	20 ⁺	—	20	25 [—]	5	—
	3	90	15	—	20	—	0
	4	25	20 ⁺	40	—	10	35 [—]
	5	—	10	—	40	—	30
	6	—	—	90	35 ⁺	30	—

Again starting from Node 1, the following path is assumed.

$$\text{Path 2} = (1-2-4-6) \quad \text{and} \quad Q_2 = \min(30, 25, 35) = 25$$

Therefore, the feasible quantity of flow through the path is 25.

$$\text{Cumulative flow, } X = X + Q_2 = 50 + 25 = 75$$

Iteration 3. Now, assign minus sign to the cells 1-2, 2-4, and 4-6 and then assign positive sign to the cells 2-1, 4-2 and 6-4. Obtain Table 5.38 of Iteration 3 by subtracting Q_2 from all the negatively signed cells and adding Q_2 to all the positively signed cells of Table 5.37.

Table 5.38 Flow Matrix of Iteration 3

		To					
		1	2	3	4	5	6
From	1	—	5	10	15 [—]	—	—
	2	45	—	20	0	5	—
	3	90	15	—	20	—	0
	4	25 ⁺	45	40	—	10	10 [—]
	5	—	10	—	40	—	30
	6	—	—	90	60 ⁺	30	—

Next, starting from Node 1 again, the following path is assumed.

$$\text{Path 3} = (1-4-6) \text{ and } Q_3 = \min(15, 10) = 10$$

Therefore, the feasible quantity of flow through the path is 10.

$$\text{Cumulative flow, } X = X + Q_3 = 75 + 10 = 85$$

Iteration 4. Now, assign minus sign to the cells 1-4 and 4-6 and then assign positive sign to the cells 4-1 and 6-4. Obtain Table 5.39 of iteration 4 by subtracting Q_3 from all the negatively signed cells and adding Q_3 to all the positively signed cells of Table 5.38.

Table 5.39 Flow Matrix of Iteration 4

		To					
		1	2	3	4	5	6
From	1	—	5 [—]	10	5	—	—
	2	45 ⁺	—	20	0	5 [—]	—
	3	90	15	—	20	—	0
	4	35	45	40	—	10	0
	5	—	10 ⁺	—	40	—	30 [—]
	6	—	—	90	70	30 ⁺	—

Next, again starting from Node 1, the following path is assumed.

$$\text{Path 4} = (1-2-5-6) \text{ and } Q_4 = \min(5, 5, 30) = 5$$

Therefore, the feasible quantity of flow through the path is 5.

$$\text{Cumulative flow, } X = X + Q_4 = 85 + 5 = 90$$

Iteration 5. Now, assign minus sign to the cells 1-2, 2-5 and 5-6 and then assign positive sign to the cells 2-1, 5-2 and 6-5. Obtain Table 5.40 of Iteration 5 by subtracting Q_4 from all the negatively signed cells and adding Q_4 to all the positively signed cells of Table 5.39.

Table 5.40 Flow Matrix of Iteration 5

		To					
		1	2	3	4	5	6
From	1	—	0	10 ⁻	5	—	—
	2	50	—	20	0	0	—
	3	90 ⁺	15	—	20 ⁻	—	0
	4	35	45	40 ⁺	—	10 ⁻	0
	5	—	15	—	40 ⁺	—	25 ⁻
	6	—	—	90	70	35 ⁺	—

Again starting from Node 1, the following path is assumed.

$$\text{Path 5} = (1-3-4-5-6) \quad \text{and} \quad Q_5 = \min(10, 20, 10, 25) = 10$$

Therefore, the feasible quantity of flow through the path is 10.

$$\text{Cumulative flow, } X = X + Q_5 = 90 + 10 = 100$$

Iteration 6. Now, assign minus sign to the cells 1-3, 3-4, 4-5 and 5-6 and then assign positive sign to the cells 3-1, 4-3, 5-4 and 6-5. Obtain Table 5.41 of Iteration 6 by subtracting Q_5 from all the negatively signed cells and adding Q_5 to all the positively signed cells of Table 5.40.

Table 5.41 Flow Matrix of Iteration 6

		To					
		1	2	3	4	5	6
From	1	—	0	0	5	—	—
	2	50	—	20	0	0	—
	3	100	15	—	10	—	0
	4	35	45	50	—	0	0
	5	—	15	—	50	—	15
	6	—	—	90	70	45	—

Now, no more flow is possible from Node 1 to Node 6. Subtract the matrix elements of Table 5.41 from the corresponding elements of Table 5.36 and write only the positive values in Table 5.42. Table 5.42 gives the final results and the corresponding flows are mapped onto different arcs in Fig. 5.37. The maximal flow from Node 1 to Node 6 is 100 units.

Table 5.42 Final Flow Matrix

		To					
		1	2	3	4	5	6
From	1	—	30	60	10	—	—
	2		—	—	25	5	—
	3			—	10	—	50
	4				—	10	35
	5					—	15
	6						—

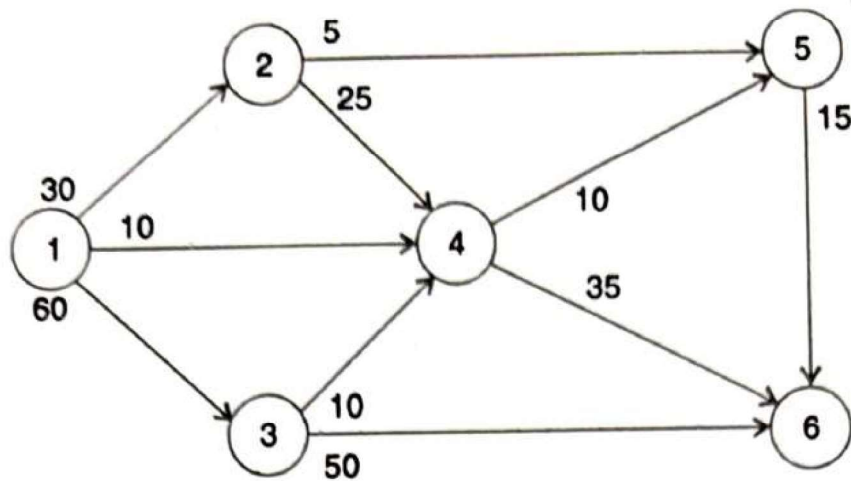


Fig. 5.37 Network showing final flows in arcs with respect to maximal flow.

QUESTIONS

- What is the shortest path problem? Give some practical applications of the shortest path problem.
- What is minimum spanning problem? What are its practical applications?
- State maximal flow problem and give its practical applications.
- Explain the steps of the following algorithms:
 - Dijkstra's algorithm,
 - Floyd's algorithm
 - PRIM algorithm
 - Kruskal's algorithm
- Explain the matrix method for the maximal flow problem.
- Consider the details of a distance network as shown below:

Arc	Distance	Arc	Distance
1-2	8	3-6	6
1-3	5	4-5	8
1-4	7	4-6	12
1-5	16	5-8	7
2-3	15	6-8	9
2-6	3	6-9	15
2-7	4	7-9	12
3-4	5	8-9	6

- Construct the distance network.
 - Find the shortest path from Node 1 to Node 9 using the systematic method.
 - Find the shortest path from Node 1 to Node 9 using Dijkstra's algorithm.
- Consider the details of a distance network as shown below:

Arc	Distance
1-2	3
1-3	8
1-4	10
2-3	4
2-4	7
3-4	2
3-5	8
4-5	6

- (a) Construct the distance network.
- (b) Apply Floyd's algorithm and obtain the final matrices, D^5 and P^5 .
- (c) Find the shortest path and the corresponding distance for each of the following:
 - (i) from Node 1 to Node 5
 - (ii) from Node 2 to Node 5.

8. Consider the details of a distance network as shown below:

Arc	Distance	Arc	Distance
1-2	6	5-6	13
1-3	7	5-8	9
1-4	10	6-7	5
2-3	8	6-8	4
2-5	4	6-9	8
3-4	6	6-10	3
3-5	11	7-9	10
3-6	3	8-10	10
3-7	5	9-10	9
4-7	7		

- (a) Construct the distance network.
 - (b) Find the minimum spanning tree using PRIM algorithm.
9. Consider the distance network which is given in Question 8 and find the minimum spanning tree using Kruskal's algorithm.
10. Consider the following details of piping network which is used to transfer oil.
- (a) Draw the flow network.
 - (b) Develop a linear programming model to determine the maximum flow from the Node 1 to the Node 6 and also the corresponding flow pattern.

Arc $i-j$	Flow		Arc $i-j$	Flow	
	f_{ij}	f_{ji}		f_{ij}	f_{ji}
1-2	20	-	3-4	13	-
1-3	25	-	3-5	10	8
2-3	5	10	4-5	15	-
2-4	9	4	4-6	30	-
2-5	15	-	5-6	25	-

11. Consider the data of a flow network as shown below:

- Draw the flow network.
- Determine the maximum flow from Node 1 to Node 6 and also the corresponding flow pattern using the matrix method.

Arc $i-j$	Flow		Arc $i-j$	Flow	
	f_{ij}	f_{ji}		f_{ij}	f_{ji}
1-2	60	10	3-4	35	-
1-3	35	25	3-5	30	28
2-3	25	20	4-5	45	-
2-4	19	24	4-6	40	-
2-5	25	30	5-6	55	-

UNIT-IV

GAME THEORY

INTRODUCTION

The term game refers to a situation of conflict and competition in which two or more competitors (or participants) are involved in decision-making in anticipation of certain outcomes over a period of time. The competitors are referred as players. A player may be an individual, a group of individuals or an organization.

Example:

- Two player game- chess.
- Pricing of products, where a firm's ultimate sales are determined not only by the price levels it selects but also by the prices its competitors set.
- Success of an advertising/marketing campaign depends largely on various types of services offered to the customers, etc.

As an area of academic study, theory of games provides a series of mathematical models that may be useful in explaining interactive decision-making concepts where two or more competitors are involved under conditions of conflict and competition.

Game theory came into existence in 20th century. However, in 1944 John Von Neumann and Oscar Morgenstern published a book "Theory of games and economic behaviour", where they discussed how businesses of all types may use this technique to determine the best strategies given a competitive business environment. Authors' approach was based on the principle of "best out of the worst".

Important definitions

Number of players: If a game involves only two players [competitors], then it is called a two-person game. However, if the number of players is more, the game is referred to as n-person game.

Gain, Loss and Value of the Game: The positive result of one player is gain and the same is considered as loss to another player.

If in a game sum of the gains to one player is exactly equal to the sum of losses to another player, so that sum of the games and losses equals to zero, then the game is said to be a *zero-sum game*. Otherwise it is said to be a *non-zero sum game*.

Payoff matrix: The payoffs in terms of gains or losses, when players select their particular strategies, can be represented in the form of a matrix, called the payoff matrix.

The rows of the payoff matrix denote player A's strategies and the columns denote player B's strategies.

Since player A is assumed to be the gainer always, so he wishes to gain as large a payoff a_{ij} as possible while player B [looser] will do his best to reach as small a value of a_{ij} as possible. If player B is gainer then player will be looser.

Player A has 'm' strategies represented by the letters: $A_1, A_2, \dots, A_m$ and player B has 'n' strategies represented by the letters: $B_1, B_2, \dots, B_n$. The number m and n need not be equal. The total

number of possible outcomes is therefore $m \times n$. Here, it is assumed that each player knows not only his own list of possible courses of action but also of his opponent.

		Courses of action (Strategies)							Probability
		S_1	S_2	S_3		S_j		S_n	
States of nature	N_1	P_{11}	P_{12}	P_{13}		P_{1j}		P_{1n}	P_1
	N_2	P_{21}	P_{22}	P_{23}		P_{2j}		P_{2n}	P_2
	.								.
	.								.
	N_i	P_{31}	P_{32}	P_{33}		P_{3j}		P_{3n}	P_i
	.								.
	N_m	P_{m1}	P_{m2}	P_{m3}		P_{mj}		P_{mn}	P_m

TWO-PERSON ZERO-SUM GAMES

A game with only two players, say player A and player B is called a *two-person zero-sum game*, if one player's gain is equal to the loss of other player so that total sum is zero.

Since the game is zero-sum, the gain of one player is equal to the loss of other and vice versa.

STRATEGY: The strategy for a player is the list of all possible actions [courses of action] that he will take for every payoff [outcome] that might arise.

- Player A's strategies, $S(A) = \{p_1, p_2, \dots, p_m\}$
- Player B's strategies, $S(B) = \{q_1, q_2, \dots, q_n\}$

Optimal Strategy

The particular strategy by which a player optimises his gains or losses without knowing the competitors strategies is called optimal strategy.

Pure Strategy

It is the decision rule which is always used by the player to select the particular strategy [course of action]. Thus, each player knows in advance of all strategies out of which he always selects only one particular strategy regardless of the other player's strategy, and the objective of the players is to maximize gains or minimize losses.

Mixed Strategy

Courses of action that are to be selected on a particular occasion with some fixed probability are called mixed strategies. Thus, there is a probabilistic situation and objective of the players is to maximize expected gains or to minimize expected losses by making choice among pure strategies with fixed probabilities.

SOLUTION OF GAMES

The games can be solved as follows:

1. Pure strategies
2. Mixed strategies

Games with saddle point [pure strategies]

- “Saddle point of a payoff matrix is an equilibrium point or common point of row and column”
- “If the maximin value equals to minimax value, then the game is said to have a saddle point”.

Rules to determine saddle point

- a. Select the minimum [lowest] element in each row of the payoff matrix and write them under ‘row minima’ heading. Then select the largest element among these elements and enclose it in a rectangle, ‘’.
- b. Select the maximum [largest] element in each column of the payoff matrix and write them under ‘column maxima’ heading. Then select the lowest element it in a circle, ‘’.
- c. Find out the element(s) which is same in the rectangle as well as circle and mark the position of such element(s) in the matrix. This element represents the value of the game and is called the saddle (or) equilibrium point.

Games saddle point [mixed strategies]

In certain cases, there is no pure strategy solution for a game, i.e., no saddle point exists. In all such cases, to solve games both the players must determine an optimal mixture of strategies to find a saddle point.

The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being selected. The strategies so determined are called mixed strategies because they are probabilistic combination of available choices by strategy.

The value of game obtained by the use of mixed strategies represents best payoff which player A can expect to win and the least which player B can lose. The expected payoff to a player in a game with arbitrary payoff matrix $[a_{ij}]$ of order $m \times n$ is defined as:

$$E(p,q) = \sum_{i=1}^m \sum_{j=1}^n p_i a_{ij} q_j$$

$$= P^T A Q \text{ [in matrix notation]}$$

where,

P = Player A's strategies, $S(A) = \{ p_1, p_2, \dots, p_m \}$

Q = Player B's strategies, $S(B) = \{ q_1, q_2, \dots, q_m \}$

Note of Book:

- $p_1 + p_2 + \dots + p_m = 1$
- $q_1 + q_2 + \dots + q_m = 1$

- A player chooses a particular strategy with particular probability can also which be interpreted as the relative frequency with a strategy is chosen from the number of strategies available to a player in a particular game.

Rules to determine saddle point

If the saddle point do not exists in a given mXn payoff matrix, then the following rules are used to reduce the matrix into 2X2.

- In a mXn payoff matrix, if each element in a row, say R_d , is less than or equal to the corresponding element in another row, say R_s , then the row R_d is said to be dominated by row R_s and therefore, row R_d can be deleted from the payoff matrix.
- In a mXn payoff matrix, if each element in a column, say C_d , is greater than or equal to the corresponding element in another column, say C_s , then the column C_d is said to be dominated by column C_s and therefore, column C_d can be deleted from the payoff matrix.
- In a mXn payoff matrix, if the domination is strict, then the rows and/or columns are deleted by average method as per the rules 'a' and 'b'.

2X2 matrix:

The mXn matrix can be reduced as 2X2 matrix and the payoff values are written as follows:

2X2 matrix

		Player B	
		B1	B2
Player A	A1	a_{11}	a_{12}
	A2	a_{21}	a_{22}

Or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Note of Book:

- In 2X2 matrix the optimal strategies for player A and B are given as follows:
Player A's strategies, $S(A) = \{ p_1, p_2 \}$
Player B's strategies, $S(B) = \{ q_1, q_2 \}$

Methods of mixed strategies

A mixed strategy game can be solved by different solution methods such as:

- Arithmetic or short-cut method
- Graphical method
- Dominance property

Arithmetic method

This method is also called *short-cut method*. It provides an easy procedure for finding optimal strategies for each player in a 2X2 matrix without saddle point.

The following steps are used under this method:

- Find the differences between the two values in the first row and put it against second row of the matrix, neglecting negative sign (if any).
- Find the differences between the two values in the second row and put it against first row of the matrix, neglecting negative (if any).
- Repeat step 'a' and 'b' for the two columns also.

Note of Book:

- The value obtained by 'swapping the differences' represents the optimal relative frequencies of play for both players' strategies.
- d. The frequencies are converted into probabilities by dividing each of them by their sum.
- e. The value of the game can be obtained as follows:

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{[a_{11} + a_{22}] - [a_{12} + a_{21}]}$$

Note of Book:

- The value of the game can also be obtained from any one of the following equations:
 - $V = a_{11}p_1 + a_{21}p_2$
 - $V = a_{12}p_1 + a_{22}p_2$
 - $V = a_{11}q_1 + a_{12}q_2$
 - $V = a_{21}q_1 + a_{22}q_2$

Note of Book:

- The player B strategies are represented in column vector.

Graphical method

The graphical method is useful for the game where the payoff matrix is of the size '2Xn' or 'mX2' i.e., the game with mixed strategies that has only two un dominated pure strategies for one of the players in the two-person zero-sum game.

Optimal strategies for both the player assign non-zero probabilities to the same number of pure strategies. Therefore, if one player has only two strategies, the other will also use the same number of strategies. Hence, this method is useful in finding out which of the two strategies can be used.

Case:1: 2Xn matrix is given:

Consider the following 2Xn payoff matrix of a game without saddle point.

		Player B				Probability
		B_1	B_2		B_n	
Player A	A_1	a_{11}	a_{12}		a_{1n}	p_1
	A_2	a_{21}	a_{22}		a_{2n}	p_2

Probability	q_1	q_2	$\dots q_n$

Player A has two strategies A_1 and A_2 with probability of their selection p_1 and p_2 , respectively, such that $p_1 + p_2 = 1$ and $p_1, p_2 \geq 0$. Now for each of the pure strategies available to player-B, expected payoff for player A would be as follows:

B's pure strategies	A's expected payoff
B_1	$a_{11}p_1 + a_{21}p_2$
B_2	$a_{12}p_1 + a_{22}p_2$
$\vdots$	$\vdots$
B_n	$a_{1n}p_1 + a_{2n}p_2$

According to the maximin criterion for mixed strategy games, player A should select the value of probability p_1 and p_2 so as to maximize his minimum expected payoffs. This may be done by plotting the straight lines representing player A's expected payoff values.

The highest point on the lower boundary of these lines will give maximum expected payoff among the minimum expected payoffs and the optimum value of probability p_1 and p_2 .

Now the two strategies of player B corresponding to those lines which pass through the maximin point can be determined. It helps in reducing the size of the game to (2X2), which can be easily solved by any of the methods discussed earlier.

Case:2: mX2 matrix is given:

Consider the following mXn payoff matrix of a game without saddle point.

		Player B		Probability
		B_1	B_2	
Player A	A_1	a_{11}	a_{12}	p_1
	A_2	a_{21}	a_{22}	p_2
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	A_m	a_{m1}	a_{m2}	p_m
Probability		q_1	q_2	

Player B has two strategies B_1 and B_2 with probability of their selection q_1 and q_2 , respectively, such that $q_1 + q_2 = 1$ and $q_1, q_2 \geq 0$. Now for each of the pure strategies available to player-A, expected payoff for player B would be as follows:

A's pure strategies	B's expected payoff
---------------------	---------------------

A_1	$a_{11}q_1 + a_{12}q_2$
A_2	$a_{21}q_1 + a_{22}q_2$
$\vdots$	$\vdots$
A_m	$a_{m1}q_1 + a_{m2}q_2$

According to the minimax criterion for mixed strategy games, player B should select the value of probability q_1 and q_2 so as to minimize his maximum expected payoffs. This may be done by plotting the straight lines representing player B's expected payoff values.

The lowest point on the upper boundary of these lines will give minimum expected payoff among the maximum expected payoffs and the optimum value of probability q_1 and q_2 .

Now the two strategies of player A corresponding to those lines which pass through the minimax point can be determined. It helps in reducing the size of the game to (2×2) , which can be easily solved by any of the methods discussed earlier.

Dominance property:

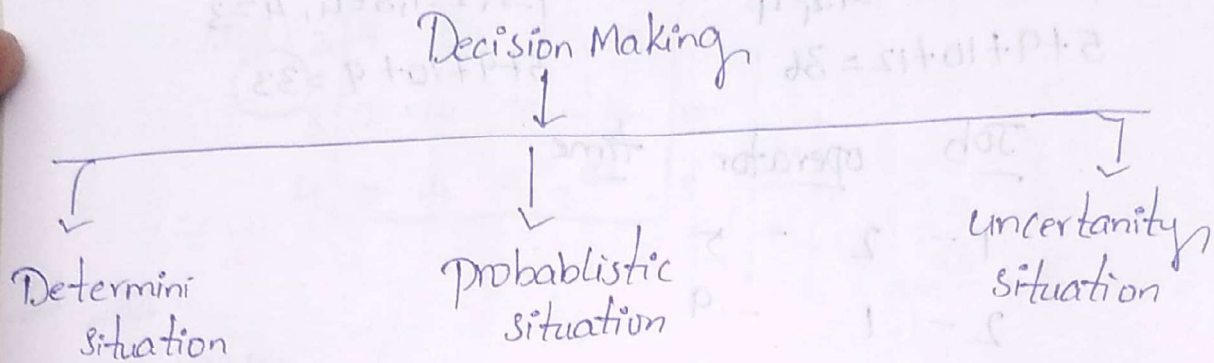
In some of the games, it is possible to reduce the size of the payoff matrix by eliminating rows (or columns) which are dominated by other rows (or columns) respectively.

Dominance property for rows: $X \leq Y$ i.e. if all the elements of a particular row X are less than or equal to the corresponding elements of another row Y then delete row X (row X is dominated by row Y). The elements of a particular row X can also be compared with an average of two or more other rows and if the elements of the row X are less than or equal to the corresponding elements after taking the average then delete row X .

Dominance property for columns: $X \geq Y$ i.e. if all the elements of a particular column X are greater than or equal to the corresponding elements of another column Y then delete column X (column X is dominated by column Y). The elements of the column X can also be compared with an average of two or more columns and if the elements of column X are greater than the corresponding elements after taking the average then delete the column X .

Game Theory

It is a mathematical model which is used for decision making.

Terminology of Game theory:

- 1) players: Player A and Player B
- 2) strategy: It is course of action taken by a player.
 → Pure strategy: The player will select only one particular strategy & ignore remaining strategy.

$$P_1 = 0, P_2 = 1, P_3 = 0$$

$$\text{Total: } 0 + 1 + 0 = 1$$

∴ sum of probability is '1'.

3) Mixed strategy:

It means, if the player follows more than one strategy.

$$P_1 = 0.65, P_2 = 0.35, P_3 = 0$$

The sum will be '1' only.

4) Pay off matrix:

		Player B		
		1	2	3
Player A	1	20	10	22
	2	30	40	38
	3	15	18	25

- matrix entries are outcomes (like 20, 10, 22, 30, 40)
- outcome is positive means it is gain to player A, lost to player 'B'.
- outcome is Negative means it is gain to player B, lost to player 'A'.

5) maximin principal:
maximizes the minimum guarantee of player A

6) Minimax principal:
minimizes the maximum losses

7) saddle point:
maximin value is equal to minimax value

8) Value of the Game:
If the game has a saddle point then the value of the cell at the saddle point is called the value of the game.

9) Two person '0' sum game: In a game with 2 players if the gain of one player is equal to the loss of another player. Then the game is called Two person '0' sum game.

Games with saddle point

Game with pure strategy:

1) Find the optimum strategies of the players in the following games

		Player B		
		1	2	3
Player A	1	25	20	25
	2	50	45	55
	3	58	40	42

Sol Given

		Player B			
		1	2	3	<u>Row min</u>
Player A	1	25	20	25	20
	2	50	45	55	<u>45</u> maximin
	3	58	40	42	40
<u>Col: max</u>		58	<u>45</u> minimax	55	

As per the pay of matrix Player A is called maximin player [Row], Player B is called minimax player

→ It means Player A is maximizing its minimum garently gains.

→ Player B is minimizing its maximum loss.

Note:

Player A = maximin = 45

Player B = minimax = 45

∴ This Game has a saddle point.

[value of Game $V = \{45\}$]

∴ Hence the Game has saddle point at the cell corresponding to row to E_1 column.

optimum probability of Player A $[P_1, P_2, P_3] = [0, 1, 0]$

B $[a_1, a_2, a_3] = [0, 1, 0]$

This game has a pure strategy.

2) Solve the matrix using game theory to find the saddle point.

		1	2	3
Player A	1	1	3	1
	2	0	-4	-3
	3	1	5	-1

Sol Given matrix

	1	2	3
1	1	3	1
2	0	-4	-3
3	1	5	-1

$$\max\{\min \text{ row}\} = \min\{\max \text{ col}\}$$

$$\max\{1, -4, -1\} = \min\{1, 5, 1\}$$

$$1 = 1$$

from the above game have saddle point is 1.

3) Solve the matrix using game theory to find out the Saddle point.

		Player B			
		1	2	3	4
Player A	1	-5	-2	0	7
	2	5	-6	-4	8
	3	4	0	2	3

Sol Given

		Player B				
		1	2	3	4	R.M
Player A	1	-5	-2	0	7	-5
	2	5	-6	-4	8	-6
	3	4	0	2	3	0
Col-max		5	0	2	8	

minimax

$$\max \{ \min \text{ row } \} = \min \{ \max \text{ col } \}$$

$$\max \{ -5, -6, 0 \} = \min \{ 5, 0, 2, 8 \}$$

$$0 = 0$$

∴ From above game have saddle optimum probability of Player A $(P_1, P_2, P_3) = (0, 0, 1)$

Player B $(a_1, a_2, a_3, a_4) = (0, 1, 0, 0)$

The value of game = 0

4) Solve the game for the given matrix to find out the saddle point

		Player B				
		1	2	3	4	5
Player A	1	3	-1	4	6	7
	2	-1	8	2	4	12
	3	16	8	6	14	12
	4	1	11	-4	2	1

Given

	1	2	3	4	5	R. Min
1	3	-1	4	6	7	-1
2	-1	8	2	4	12	-1
3	16	8	6	14	12	6 maximin
4	1	11	-4	2	1	-4
Col-max	16	11	6	14	12	

minimax

$$\max(\min \text{row}) = \min(\max \text{column})$$

$$\max(-1, -1, 6, -4) = (16, 11, 6, 14, 12)$$

$$6 = 6$$

∴ From the above matrix have a saddle point.
optimum condition for Player A = (0, 0, 1, 0)

$$\text{Player B} = (0, 0, 1, 0, 0)$$

The value of the game = 6

⑤ solve the following game, the given matrix is saddle point (or) not?

		Player B				
		1	2	3	4	5
Player A	1	10	5	3	9	1
	2	9	6	5	7	8
	3	3	5	4	6	9
	4	6	7	3	3	2

sol Given

The pay of matrix

	1	2	3	4	5	R. Min
1	10	5	3	9	1	1
2	9	6	5	7	8	5 maximin
3	3	5	4	6	9	3
4	6	7	3	3	2	2
Col-max	10	7	5	9	9	

minimax

$$\max(\min \text{ row}) = \min(\max \text{ column})$$

$$\max(1, 5, 3, 2) = \min(10, 7, 5, 9, 9)$$

$$5 = 5$$

$\therefore$ The value of the game = 5

$\therefore$ From the above matrix have saddle point the optimum probability of player A = $\{0, 1, 0, 0\}$
player B = $\{0, 0, 1, 0\}$

⑥ Solve the following given matrix with using game theory to find out the saddle point.

	player B			
	1	2	3	4
1	1	7	3	4
2	5	6	4	5
3	7	2	1	3

Sol Given

		player B			
		1	2	3	4
player A	1	1	7	3	4
	2	5	6	4	5
	3	7	2	1	3
	<u>101. max</u>	7	7	<u>4</u>	5
				minmax	

$$\max(\min \text{ row}) = \min(\max \text{ col.})$$

$$\max(1, 4, 1) = \min(7, 7, 4, 5)$$

$$4 = 4$$

From the above matrix have a saddle point optimum probability of player A = $(0, 1, 0)$
Player B = $(0, 0, 1, 0)$

The value of the game = 4.

Games without saddle point:

Consider the following pay off matrix with respect to player A & solving optimally.

	1	2
Player A 1	9	7
2	5	11

sol Given

	1	2	R.m
1	9	7	Ⓐ maxmin
2	5	11	5

Column ① 11
min max

$$\max(\text{Row min}) = \min(\text{column max})$$

$$\max(7, 5) = \min(9, 11)$$

$7 \neq 9$ [Arithmetic method]

$\therefore$ This game has no saddle point

		Player B	oddment	probability
Player A	9	7	6	$P_1 = \frac{6}{6+2} = \frac{6}{8} = \frac{3}{4}$
	5	11	2	$P_2 = \frac{2}{6+2} = \frac{1}{4}$
	4	4		

$$q_1 = \frac{4}{4+4} = \frac{1}{2}, q_2 = \frac{4}{4+2} = \frac{1}{2}$$

$\therefore$ value of the game, we have many formulas for calculating value of the game i.e.,

$$V = \frac{(9 \times 6) + (5 \times 2)}{6+2} = \frac{64}{8} = 8$$

(or)

$$V = \frac{(7 \times 6) + (11 \times 2)}{6+2} = 8$$

(or)

$$V = \frac{(9 \times 4) + (7 \times 4)}{4+4} = 8$$

(or)

$$V = \frac{(5 \times 1) + (1 \times 4)}{1+4} = 8$$

Hence the standards of player A $(\frac{3}{4}, \frac{1}{4})$
 player B $(\frac{1}{2}, \frac{1}{2})$

The value of the game = 8

② Solve the following game.

5	1
3	4

Given

5	1	1	$P_1 = \frac{1}{5}$
3	4	4	$P_2 = \frac{4}{5}$
3	2		

$$q_1 = \frac{3}{5} \quad q_2 = \frac{2}{5}$$

The value of the game

$$V = \frac{5+12}{5} = \frac{17}{5}$$

The optimum strategy of

$$\text{Player A} = (\frac{1}{5}, \frac{4}{5})$$

$$\text{Player B} = (\frac{3}{5}, \frac{2}{5})$$

③ Solve the game without saddle point

1	7
6	2

Given

1	7	4	$P_1 = \frac{4}{10}$
6	2	6	$P_2 = \frac{6}{10}$
5	5		

$$q_1 = \frac{5}{10} \quad q_2 = \frac{5}{10}$$

$$(7-1) = 6$$

$$(6-2) = 4$$

The optimum probability

$$\text{Player A} = (\frac{4}{10}, \frac{6}{10})$$

$$\text{Player B} = (\frac{5}{10}, \frac{5}{10})$$

The value of the game $V = \frac{(1 \times 4) + (6 \times 6)}{4+6}$

$$V = \frac{4+36}{10} = \frac{40}{10} = 4$$

Dominance property: three by three ($\frac{3}{3}$)
 consider the four by four ($\frac{4}{4}$) game played by
 Player A & Player B. & solve it optimally.

		I	II	III
Player A	I	20	-20	50
II	-25	25	-25	
III	20	-50	50	

row - least delete
 column - high - delete

Sol Given

The pay of matrix with respect to player 'A'

	I	II	III	R _{min}
I	20	-20	50	<u>-20</u> maxmin
II	-25	25	-25	-25
III	20	-50	50	-50
col-max	<u>20</u>	25	50	

minimax value

→ The game has not saddle point.

check the dominance property row is dominated
 by row I & Hence row III is to be deleted. The
 resultant matrix after deleting row III.

Pay of matrix after deleting row III.

	I	II	III
I	20	-20	50
II	-25	25	-25
III	20	-50	50

The column III is dominated by column I & Hence,
 column three (III) is to be deleted. The resultant
 matrix after deleting column (III).

Pay of matrix after deleting column III

20	-20	50
-25	25	-25

Now calculate the oddments & probability.

20	-20	50
-25	25	40
45	45	

oddments/Probability

$$P_1 = \frac{50}{40+50} = \frac{50}{90} = \frac{5}{9}$$

$$P_2 = \frac{40}{50+40} = \frac{40}{90} = \frac{4}{9}$$

(row)

$$\Rightarrow 20 - (-20) = 20 + 20 = 40$$

$$\Rightarrow 25 - (-25) = 25 + 25 = 50$$

(col)

$$\Rightarrow 20 - (-25) = 20 + 25 = 45$$

$$\Rightarrow 25 - (-25) = 25 + 25 = 45$$

$$a_1 = \frac{45}{45+45} = \frac{45}{90} = \frac{1}{2}$$

$$a_2 = \frac{45}{45+45} = \frac{45}{90} = \frac{1}{2}$$

Value of game:

$$V = \frac{(20 \times 50) + (-25 \times 40)}{50 + 40}$$

$$= \frac{1000 - 1000}{90} = 0$$

Hence the strategies of Player A & Player B & value of the game

$$A \left(\frac{5}{9}, \frac{4}{9}, 0 \right)$$

$$B \left(\frac{1}{2}, \frac{1}{2}, 0 \right)$$

$$V = 0$$

20	-20	50	I
-25	25	-25	II
50	-50	0	III

2) The Pay off matrix of the player A is given

2	4	8	3	4
5	6	7	3	8
6	7	8	9	7
4	2	4	8	3

using dominance property obtain the optimum strategy for both the players & determine the value of game?

Sol Given

	I	II	III	IV	V	Row min
I	2	4	8	3	4	2
II	5	6	7	3	8	3
III	6	7	8	9	7	6 (highest)
IV	4	2	4	8	3	2
Col-max	6 (least)	7	8	9	8	

$$\text{maximin} = \text{minimax}$$

$$6 = 6$$

$\therefore$ The game has a saddle point value of the game $V = 6$

$$\text{Player A} (0, 0, 1, 0) = 1$$

$$\text{Player B} (1, 0, 0, 0, 0) = 1$$

	I	II	III	IV	V
I	2	4	8	3	4
II	5	6	7	3	8
III	6	7	8	9	7
IV	4	2	4	8	3

	I	II	III	IV	V
II	5	6	7	3	8
III	6	7	8	9	7

	I	IV
II	5	3
III	6	9

	I	IV
III	6	9

$V=6$

Player A (0, 0, 1, 0)

Player B (1, 0, 0, 0, 0)

3) The following table represents the pay of matrix with respect to Player A. solve it optimally using dominance property.

player B

	I	II	III	IV	V
I	4	6	5	10	6
II	7	8	5	9	10
III	8	9	11	10	9
IV	6	4	10	6	4

Sol

	I	II	III	IV	V	R.M
I	4	6	5	10	6	4
II	7	8	5	9	10	5
III	8	9	11	10	9	8
IV	6	4	10	6	4	4
Col max	8	9	11	10	10	
minimax						

maximin

$$\text{minimax} = \text{maximin}$$

$$8 = 8$$

Value of the game $v = 8$

Player A $(0, 0, 1, 0)$

Player B $(1, 0, 0, 0, 0)$

This has a saddle point, it is pure strategy.

	I	II	III	IV	V
I	4	6	5	10	6
II	7	8	5	9	10
III	8	9	11	10	9
IV	6	4	10	6	4

	I	II	III	IV	V
II	7	8	5	9	10
IV	8	9	11	10	9

	I	III
II	7	5
IV	8	11

$$v = 8$$

Player A $= (0, 0, 1, 0)$

Player B $= (1, 0, 0, 0, 0)$

4) dominance property in mixed start
 consider the 3×5 game played by player A & B
 solve it optimally

	Player B					
Player A	I	2	5	10	7	2
	II	3	3	6	6	4
	III	4	4	8	12	1

Sol Given

	I	II	III	IV	V	Row min
I	2	5	10	7	2	2
II	3	3	6	6	4	3
III	4	4	8	12	1	1
	4	5	10	12	2	minmax

maximin $\neq$ minimax

$$3 \neq 4$$

$\Rightarrow$ The game by without a saddle point

pay of matrix after deleting column.

2	5	10	7	2
3	3	6	6	4
4	4	8	12	1

$\Rightarrow$ The also again pay of matrix is after deleting row.

2	2
3	4
4	1

	I	II	addments	Probability
II	3	4	3 (4-3=1)	$P_1 = \frac{3}{4}$
III	4	1	1 (4-1)=3	$P_2 = \frac{1}{4}$
	3	1		

$$a_1 = \frac{3}{4} \quad a_2 = \frac{1}{4}$$

$$\text{value of game (V)} = \frac{(3 \times 3) + (4 \times 1)}{(3+1)} = \frac{13}{4}$$

The optimal solution of problem is

$$A \left(0, \frac{3}{4}, \frac{1}{4} \right) \quad B = \left(\frac{3}{4}, 0, 0, 0, \frac{1}{4} \right)$$

$$V = \frac{13}{4}$$

5) solve the following game $\frac{4}{4}$ using dominance property.

	I	II	III	IV
Player A				
I	6	2	4	8
II	2	-1	1	12
III	2	3	3	9
IV	5	2	6	10

Sol Given

	<u>R.M</u>				
	6	2	4	8	2
	2	-1	1	12	-1
	2	3	3	9	② minmaxmin
	5	2	6	10	2
max 6	③	6	2		

minimax

maxmin $\neq$ minimax

$$2 \neq 3$$

The game without saddle point

6	2	4	8
2	-1	1	12
2	3	3	9
5	2	6	10

6	2
2	-1
2	3
5	2

	I	II	oddments	P
I	6	2	1	$\frac{1}{5}$
III	2	3	4	$\frac{4}{5}$
	1	4		
	$\frac{1}{5}$	$\frac{4}{5}$		

$$\therefore V = \frac{(6 \times 1) + (2 \times 4)}{1 + 4} = \frac{14}{5}$$

$$\text{Player A} = P\left(\frac{1}{5}, 0, \frac{4}{5}, 0\right)$$

$$B = P\left(\frac{1}{5}, \frac{4}{5}, 0, 0\right)$$

$$V = \frac{14}{5}$$

6) solve the following problem (a) game $\frac{3}{4}$ using dominance property?

	1	2	3	4
1	5	-3	3	4
2	-4	5	4	5
3	4	-4	-3	3

sol Given

					<u>Rm</u>	
	5	-3	3	4	(-3)	maxmin
	-4	5	4	5	-4	
	4	-4	-3	3	-4	
<u>max</u>	5	5	(4)	5		
			minimax			

maxmin = minimax

$$-3 \neq 4$$

The game without saddle point

5	-3	3	4
-4	5	4	5
4	-4	-3	3

5	-3	3
-4	5	4
4	-4	-3

5	-3	3	4
-4	5	4	5

The above matrix are not possible moves, now consider the averages of any 2 rows

5	-3	3
-4	5	4

$$\text{Avg: } \frac{5+(-3)}{2} = \frac{5-3}{2} = \frac{2}{2} = 1$$

$$\frac{-4+5}{2} = \frac{1}{2} = \frac{1}{2}$$

	I	II	odd Prob
iv)	5	-3	9
II	-4	5	8
	$\frac{8}{17}$	$\frac{9}{17}$	
	$\frac{8}{17}$	$\frac{9}{17}$	

The value of game is 'V'

$$V = \frac{(5 \times 9) + (-4 \times 8)}{17}$$

$$= \frac{45 - 32}{17}$$

$$= \frac{13}{17}$$

$$\text{Player B} = \left(\frac{8}{17}, \frac{9}{17}\right)$$

The value of game $V = \frac{13}{17}$
optimally strategy of Player A = $\left(\frac{9}{17}, \frac{8}{17}\right)$

⑦ Consider the 4 by 4 following game with the pay off matrix of the player A solve it optimally.

		Player B			
		1	2	3	4
Player A	1	3	2	4	0
	2	3	4	2	4
	3	4	2	4	1
	4	3	4	3	4

⑧ Given

3	2	4	0	Row
3	4	2	4	6
4	2	4	1	2
3	4	3	4	1
③ max mini				

Col. max 4 4 4 4

$$\min \{ \text{Col. max} \} = \max \{ \text{min row} \}$$

$$\min \{ 4, 4, 4, 4 \} = \max \{ 0, 2, 1, 3 \}$$

$4 \neq 3$
 $\therefore$ The game have no saddle point

i)

3	2	4	0
3	4	2	4
4	2	4	1
3	4	3	4

(ii)

3	4	2	4
4	2	4	1
3	4	3	4

(iii)

4	2	4
2	4	1
4	3	4

(iv)

2	4	1
4	3	4

(v)

2	4	1
2	3	4
3	1	

 address | Probability
 $P_1 = \frac{1}{4}$
 $P_2 = \frac{3}{4}$

$$q_1 = \frac{3}{4}, q_2 = \frac{1}{4}$$

The game of value is

$$V = \frac{4+9}{4} = \frac{13}{4}$$

The optimum strategy of Player A = $\{0, 0, \frac{1}{4}, \frac{3}{4}\}$

Player B = $\{0, 0, \frac{3}{4}, \frac{1}{4}\}$

$\therefore$ The value of the game = $\frac{13}{4}$

	1	2	3	4
1	4	2	5	8
2	6	7	1	9

	1	2	3	4
1	4	2	5	8
2	6	7	1	9

The game has without saddle point

Now we can check the dominance property

	1	2
1	4	2
2	6	7

Graphical method:

I) Two by n games:
Two rows & n columns.

II) M by Two games:
M rows & Two columns

$2 \times n$
 $m \times 2$

Graphical method for Two by n games

1) consider the pay of matrix of player A as show below
& solve it optimally using Graphical method?

		Player B				
		1	2	3	4	5
Player A	1	-4	2	5	-6	6
	2	3	-9	7	4	8

Sol Given

The pay of matrix is

	1	2	3	4	5	<u>R.M</u>	
1	-4	2	5	-6	6	<u>-6</u>	maximin
2	3	-9	7	4	8	-9	
	3	<u>2</u>	7	4	8		minimax

$\therefore$ maximin $\neq$ minimax

$$-6 \neq 2$$

$\therefore$ The game has without saddle point.

$\Rightarrow$ Now we can check the dominance property.

mixed strategy.

		Player B				
		1	2	3	4	5
Player A	1	-4	2	5	-6	6
	2	3	-9	7	4	8

$\Rightarrow$

		1	2	4
1		-4	2	-6
2	3	-9	4	

$\therefore$ It is mixed strategy.

Player A

	1	2	4	
1	-4	2	-6	x
2	3	-9	4	$1-x$

Expected Pay off function & gain of players A

B's alternatives	A's Expected pay of function	A's Expected gain	
		$x=0$	$x=1$
1	$(-4x) + 3x(1-x)$ $= -4x + 3 - 3x$ $= -7x + 3$	3	-4
2	$2x - 9(1-x) = 2x - 9 + 9x$ $= 11x - 9$	-9	$11(1) - 9 = 2$
4	$-6x + 4(1-x)$ $= -6x + 4 - 4x$ $= -10x + 4$	4	$4 - 10(1) = -6$

→ From the graph (B_2 & B_4) are the Intersection points & the Resultant matrix is

	2	4	oddments	Probability
1	2	-6	13	$P_1 = \frac{13}{21}$
2	-9	4	8	$P_2 = \frac{8}{21}$
	10	11		
	$q_1 = \frac{10}{21}$	$q_2 = \frac{11}{21}$		

The value of the game V is

$$V = \frac{20 - 72}{21}$$

$$= \frac{-46}{21}$$

∴ The value of the game is $-\frac{46}{21}$

Graphical method

Date :

-4	2	5	-6	6
3	-9	7	4	8

Page No.:

②

$x=0$

$x=1$

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①

②

SAIRAM

The optimum strategy of Player A = $(\frac{13}{21}, \frac{8}{21})$

Player B = $(0, \frac{10}{21}, 0, \frac{11}{21}, 0)$

② consider the pay off matrix & solve it using graphical method

	1	2	3	4	5
1	3	0	6	-1	7
2	-1	5	-2	2	1

Sol Given

	1	2	3	4	5	R.M
1	3	0	6	-1	7	-1
2	-1	5	-2	2	1	-2

Col max 3 5 6 7
min max

maximin $\neq$ minimax

$-1 \neq 2$

$\therefore$ The game has no saddle points

Then we go to the dominance property

	1	2	3	4	5
1	3	0	6	-1	7
2	-1	5	-2	2	1

	1	3	4
1	3	6	-1
2	-1	-2	2

Expected pay off function & gain player-A

B's alternates	A's expected pay of function	A's expected gain	
		$x > 0$	$x = 1$
1	$-(3 \times x) + (-1)(1-x)$ $= 3x - 1 + x$ $= 4x - 1$	$4(0) - 1$ $= -1$	$4(1) - 1$ $4 - 1$ $= 3$
3	$(6 \times x) + (-2)(1-x)$ $= 6x - 2 + 2x$ $= 6x - 2 + 2x$ $= 8x - 2$	$8(0) - 2$ $0 - 2$ $= -2$	$8(1) - 2$ $8 - 2$ $= 6$
4	$(-1 \times x) + 2(1-x)$ $= -1x + 2 - 2x$ $= -3x + 2$	$-3(0) + 2$ $= 0 + 2$ $= 2$	$-3(1) + 2$ $= -3 + 2$ $= -1$

From the above graph (B, B_u) are the intersection points then the resultant matrix are:

	1	4	address	Probability
1	3	-1	3	$P_1 = \frac{3}{3+4} = \frac{3}{7}$
2	-1	2	4	$P_2 = \frac{4}{3+4} = \frac{4}{7}$
	3	4		

$$q_1 = \frac{3}{4+3} = \frac{3}{7} \quad q_2 = \frac{4}{3+4} = \frac{4}{7}$$

The value of the game is

$$V = \frac{(3 \times 3) + ((-1) \times 4)}{3+4}$$

$$= \frac{9 - 4}{7} = \frac{5}{7}$$

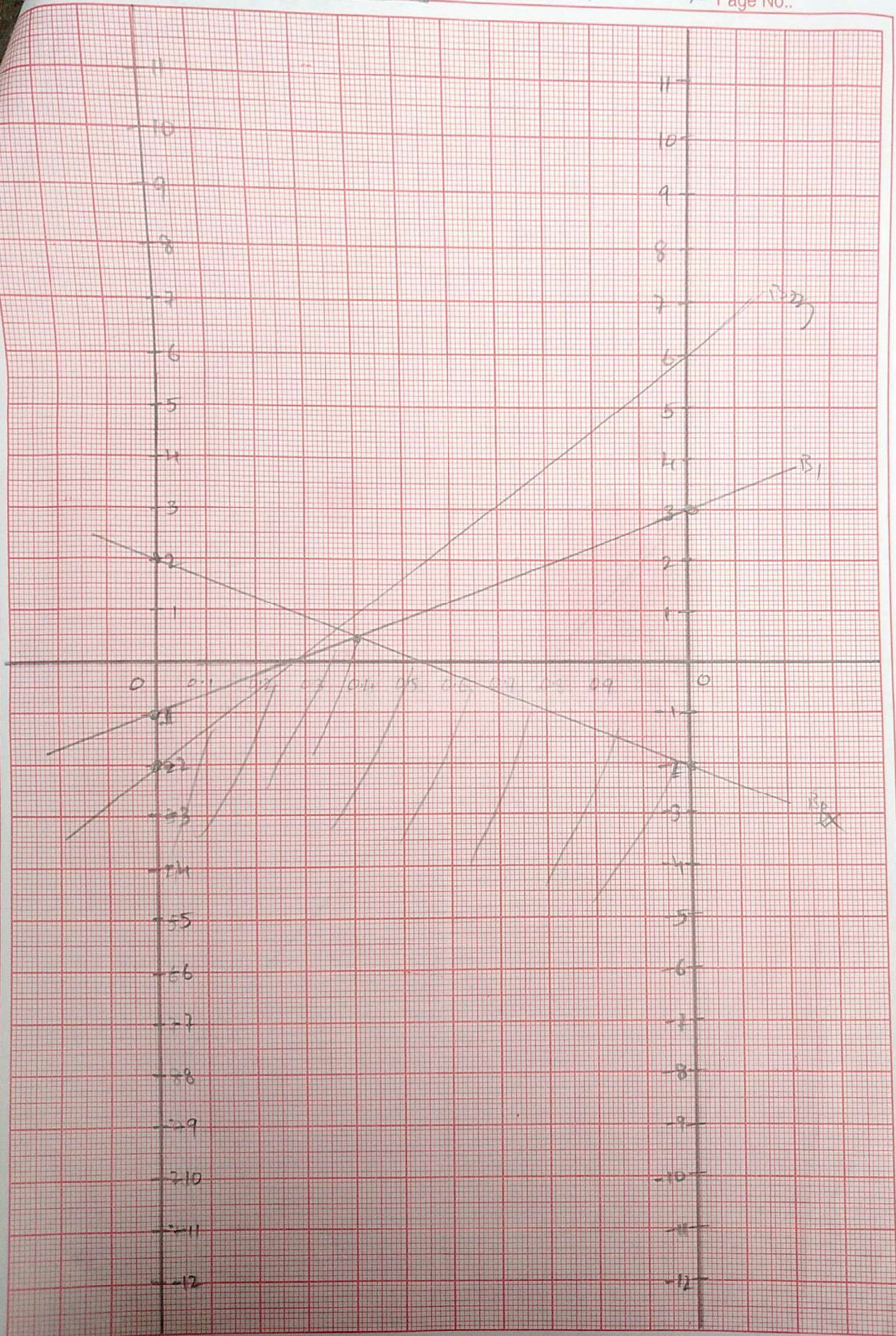
The value of game is $\frac{5}{7}$

The optimum strategy of player A = $(\frac{3}{7}, \frac{4}{7})$

Player B = $(\frac{3}{7}, \frac{4}{7})$

graphical $\begin{bmatrix} 3 & 0 & 6 & -1 & 7 \\ -1 & 5 & -2 & 2 & 1 \end{bmatrix}$
 $x=0$ method

Page No.:



3) Consider the pay off matrix $\frac{2}{5}$ two by five & solve it using Graphical method.

	I	II	III	IV	V
I	3	6	8	4	4
II	-7	4	2	10	2

Sol Given

	I	II	III	IV	V	R.M
I	3	6	8	4	4	3 maxmin
II	-7	4	2	10	2	-7
Column	3	6	8	10	4	minimax

$$\text{maximin} = \text{minimax}$$

$$3 = 3$$

$\therefore$ The given matrix has a saddle point

The pay of matrix:

	I	II	III	IV	V
I	3	6	8	4	4
II	-7	4	2	10	2

	I
I	$3x$
II	$-7(1-x)$

expected pay off function & gain player A

B's alternate	A's expected pay of matrix function	A's expected gain	
		$x=0$	$x=1$
I	$3x + -7(1-x)$ $3x - 7 + 7x$ $= 10x - 7$	$10(0) - 7$ $0 - 7$ $= -7$	$10(1) - 7$ $10 - 7$ $= 3$

$$\begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

$\therefore$ The value of game $V = 3$

Graphical method

2) $\begin{array}{|c|} \hline 3 \ 6 \ 8 \ 4 \ 4 \\ \hline 7 \ 4 \ 2 \ 10 \ 2 \\ \hline \end{array}$

Page No.:

$x=0$

11
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9

11
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12

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∴ The optimum strategy, Player A = (1, 0)
Player B = (1, 0, 0, 0, 0)

m by 2 Game

→ m Row & 2 columns

→ Graphical Method for m by 2 Game.

1) consider the pay off matrix of player A & solve it optimally using Graphical Method.

		1	2	Player B
Player A	1	6	-7	
	2	1	3	
	3	3	1	
	4	5	-1	

Sol Given

	1	2	Row
1	6	-7	-7
2	1	3	① maximin
3	3	1	① maximin
4	5	-1	-1
Column	6	③ minimax	

maximin $\neq$ minimax
 $1 \neq 3$

∴ The game has without saddle point. $\Rightarrow$

$\Rightarrow$ The dominance property has applied

^{above} the matrix, therefore the columns & rows are not deleted. Then we go to next step.

Expected pay off function & expected gain of Player B.

A's alternate	B's expected pay off function	B's expected gain	
		y=0	y=1
1	$6(y) - 7(1-y)$	$13(0) - 7$	$13(1) - 7$
2	$= 6y - 7 + 7y$	$-0 - 7$	$13 - 7$
3	$= 13y - 7$	$= -7$	$= 6$
2	$y + 3(1-y)$	$= -2y + 3$	$= -2(1) + 3$
4	$= y + 3 - 3y$	$= -2(0) + 3$	$= -2 + 3$
	$= -2y + 3$	$= 3$	$= 1$
3	$= 3y + 1(1-y)$	$= 2(0) + 1$	$= 2(1) + 1$
	$= 3y + 1 - y$	$= 1$	$= 2 + 1$
	$= 2y + 1$		$= 3$
4	$= 5y + -1(1-y)$	$= 6(0) - 1$	$= 6(1) - 1$
	$= 5y - 1 + y$	$= -1$	$= 6 - 1$
	$= 6y - 1$		$= 5$

Resultant Matrices

	1	2	odd
2	1	3	6
4	5	-1	2
	4	4	

Prob (2,4)

$$P_1 = \frac{6}{6+2} = \frac{6}{8} = \frac{3}{4}$$

$$P_2 = \frac{2}{6+2} = \frac{2}{8} = \frac{1}{4}$$

$$3 - 1 = 2$$

$$5 - (-1) = 6$$

$$3 - (-1) = 4$$

$$q_1 = \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2}$$

$$q_2 = \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2}$$

The value of game

$$V = \frac{(1 \times 6) + (5 \times 2)}{6 + 2}$$

$$= \frac{6 + 10}{8}$$

$$= \frac{16}{8}$$

$$V = 2$$

The value of game is 2

Hence the strategies of A & B of a game

$$A = \left[0, \frac{3}{4}, 0, \frac{1}{4}\right]$$

$$B = \left[\frac{1}{2}, \frac{1}{2}\right]$$

4) Consider the pay off matrix of Player A as shown in below table & solve it optimally using Graphical Method.

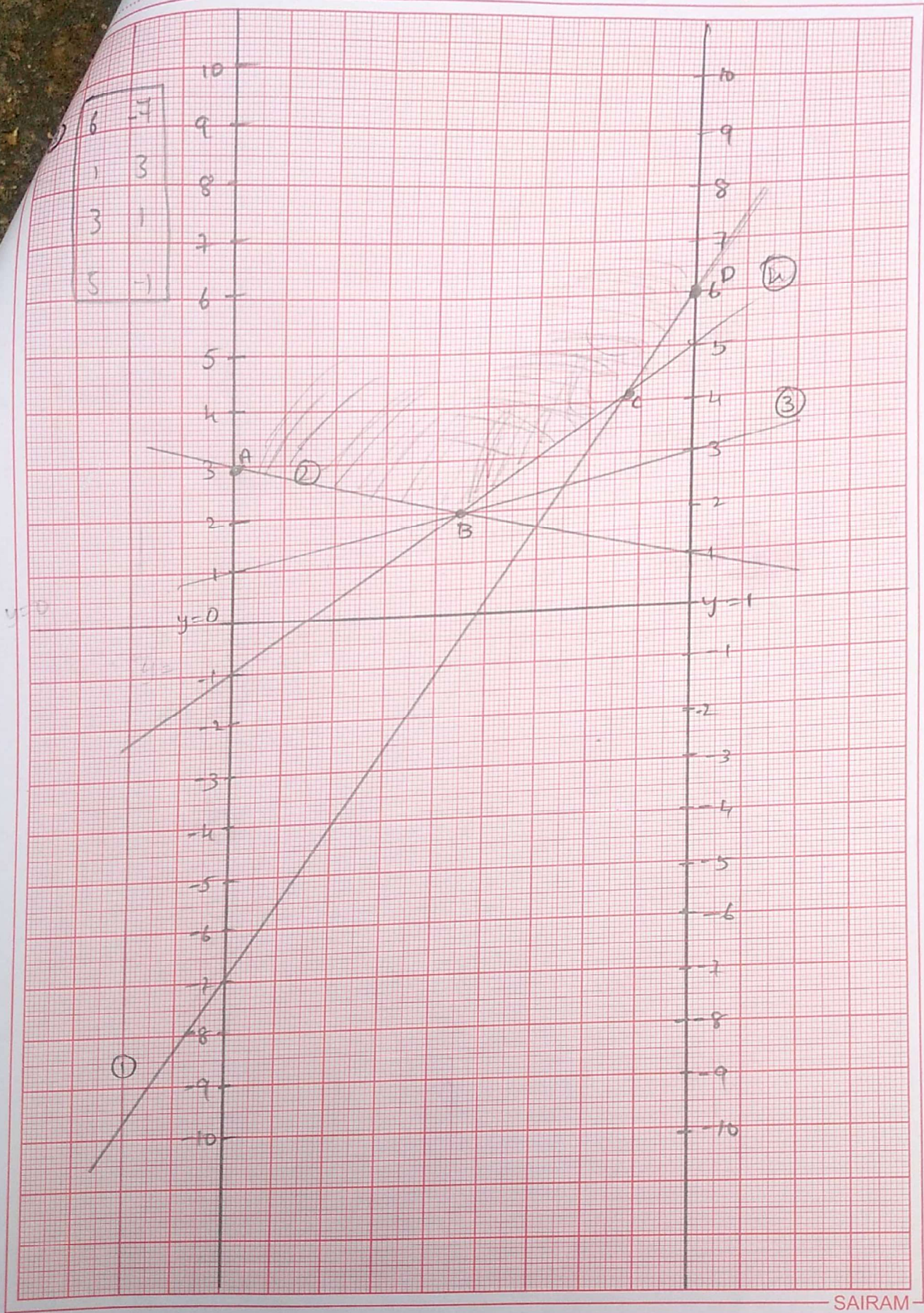
		Player B	
		1	2
Player A	1	1	3
	2	3	1
	3	5	-1
	4	6	-6

Sol Given

	1	2	R.M
1	1	3	1
2	3	1	① maximin
3	5	-1	-1
4	6	-6	-6
	6	③	minimax

Col. max

6	-4
1	3
3	1
5	-1



The value of game

$$V = \frac{(1 \times 6) + (5 \times 2)}{6 + 2}$$

$$= \frac{6 + 10}{8}$$

$$= \frac{16}{8}$$

$$V = 2$$

The value of game is 2

Hence the strategies of A & B of a game

$$A = \left[0, \frac{3}{4}, 0, \frac{1}{2}\right]$$

$$B = \left[\frac{1}{2}, \frac{1}{2}\right]$$

4) consider the pay off matrix of player A as shown in below table. & solve it optimally using graphical method.

		Player B	
		1	2
Player A	1	1	3
	2	3	1
	3	5	-1
	4	6	-6

Sol

Given

	1	2	R.M
1	1	3	1
2	3	1	① maximin
3	5	-1	-1
4	6	-6	-6
Col. max	6	③	minimax

Expected pay off function & expected gain of Player

A's alternate	B's expected pay off function	B's expected gain	
		$y=0$	$y=1$
1	$1y + 3(1-y)$ $= y + 3 - 3y$ $= -2y + 3$	$-2(0) + 3$ $0 + 3$ $= 3$	$-2(1) + 3$ $-2 + 3$ $= 1$
2	$3y + 1(1-y)$ $3y + 1 - y \Rightarrow 2y + 1$	$2(0) + 1$ $= 0 + 1$ $= 1$	$2(1) + 1$ $2 + 1$ $= 3$
3	$5y + -1(1-y)$ $5y - 1 + y \Rightarrow 6y - 1$	$6y - 1 \Rightarrow 6(0) - 1$ $0 - 1$ $= -1$	$6(1) - 1$ $6 - 1$ $= 5$
4	$6y + -6(1-y)$ $6y - 6 + 6y \Rightarrow 12y - 6$	$12(0) - 6$ $0 - 6$ $= -6$	$12(1) - 6$ $12 - 6$ $= 6$

Resultant matrixes

$5 - (-1)$
 $3 - (-1)$

	1	2	odd
1	1	3	6
3	5	-1	2
	4	4	

$P_1 = \frac{6}{6+2} = \frac{6}{8} = \frac{3}{4}$
 $P_2 = \frac{2}{6+2} = \frac{2}{8} = \frac{1}{4}$

$$q_1 = \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2} \quad q_2 = \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2}$$

$\therefore$ The value of game $\frac{(1 \times 6) + (5 \times 4)}{6+2}$

$$= \frac{6+20}{8}$$

$$= \frac{26}{8}$$

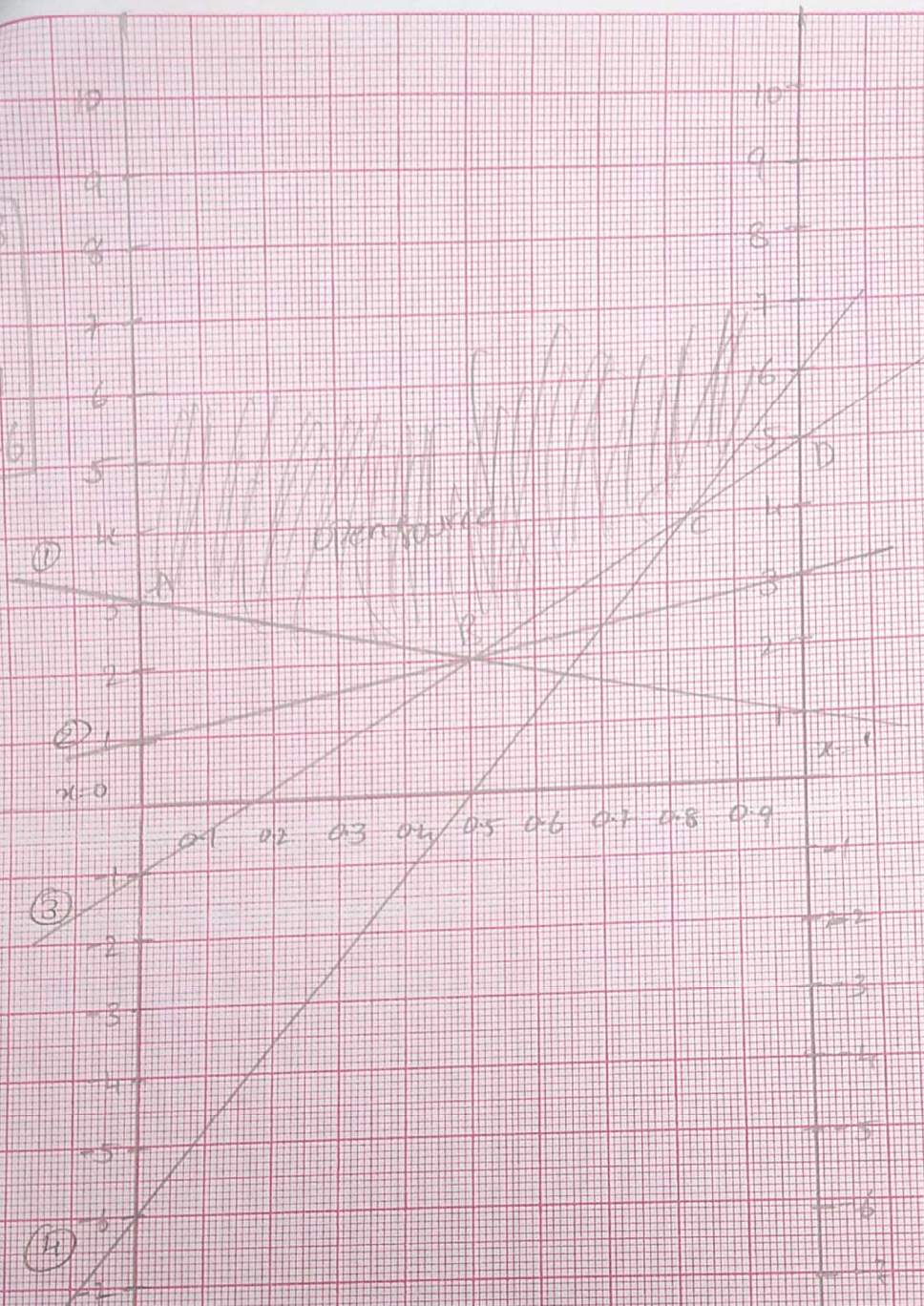
$$= \frac{13}{4}$$

$$= 3.25$$

$\therefore$ The value of game is 3.25

④

1	3
3	1
5	-1
4	-6



A_1	$a_{11}q_1 + a_{12}q_2$
A_2	$a_{21}q_1 + a_{22}q_2$
$\vdots$	$\vdots$
A_m	$a_{m1}q_1 + a_{m2}q_2$

According to the minimax criterion for mixed strategy games, player B should select the value of probability q_1 and q_2 so as to minimize his maximum expected payoffs. This may be done by plotting the straight lines representing player B's expected payoff values.

The lowest point on the upper boundary of these lines will give minimum expected payoff among the maximum expected payoffs and the optimum value of probability q_1 and q_2 .

Now the two strategies of player A corresponding to those lines which pass through the minimax point can be determined. It helps in reducing the size of the game to (2×2) , which can be easily solved by any of the methods discussed earlier.

Dominance property:

In some of the games, it is possible to reduce the size of the payoff matrix by eliminating rows (or columns) which are dominated by other rows (or columns) respectively.

Dominance property for rows: $X \leq Y$ i.e. if all the elements of a particular row X are less than or equal to the corresponding elements of another row Y then delete row X (row X is dominated by row Y). The elements of a particular row X can also be compared with an average of two or more other rows and if the elements of the row X are less than or equal to the corresponding elements after taking the average then delete row X .

Dominance property for columns: $X \geq Y$ i.e. if all the elements of a particular column X are greater than or equal to the corresponding elements of another column Y then delete column X (column X is dominated by column Y). The elements of the column X can also be compared with an average of two or more columns and if the elements of column X are greater than the corresponding elements after taking the average then delete the column X .

UNIT-V

QUEUEING THEORY

INTRODUCTION

A common situation occurring or happening in every life is that of queuing or waiting in a line.

Queuing [waiting line] are usually seen at bus stops, ticket booths, doctor's clinics, bank counters, and traffic lights and so on.

Queues are also found in workshops where the machines wait to be repaired, in warehouse where items are wait to b used, incoming calls wait to mature in the telephone exchange, trucks wait to be unloaded, airplanes wait either to takeoff or land and so on.

These problems arising due to:

- a. Too much demand on the facilities.
- b. Number of customers exceeds to the number of service facilities.
- c. Service facilities do not work efficiently.
- d. Server takes more time than prescribed time to serve a customer.

Queuing theory can be applied to a variety of operational situations where it is not possible to predict accurately the arrival rate (or time) of customers and service rate (or time) of service facility or facilities.

In practical, it can be used to determine the level of service [either the service rate or the number of service facilities] that balances the following two conflicting costs:

- a. Cost of offering the service.
- b. Cost incurred due to delay in offering service.

The first cost is associated with the service facilities and their operation, and the second represents the cost of customer's waiting time.

Many practical situations in which study of queuing theory can provide solution to waiting line problems are listed in the following table:

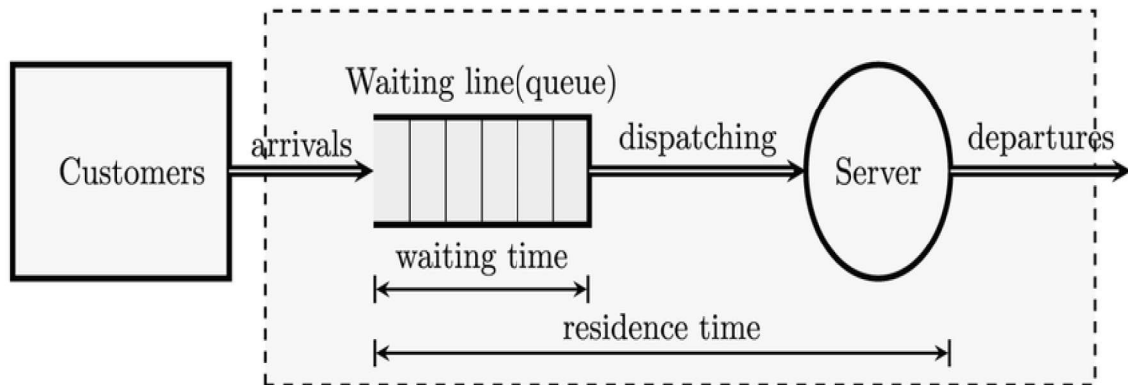
Examples of queues

Station	Customers	Service facility
1. Petrol station	Automobiles	Petrol pumps
2. Hospital	Patients	Doctors/Nurses/Rooms
3. Post office	Letters	Sorting system
4. Airport	Aircraft	Runways
5. Job interview	Applicants	Interviewers

QUEUEING STRUCTURE/SYSTEM

A queuing system is composed of the following components or parts:

1. Calling population/input source
2. Queuing process/arrival process
3. Queue discipline
4. Service process



COMPONENTS OF QUEUING STRUCTURE

Calling population or Input source

Customers are defined as those in need of service. Customers requiring service are generated at different times by “calling population”, commonly known as “input source”.

The arrivals or inputs to the system are characterised by:

- a. Size of calling population
- b. Behaviour of the arrivals
- c. Pattern of arrivals at the system

Size of calling population

The size of calling population whether it is homogeneous or consists of several subpopulations is considered to be either finite (limited) or infinite (unlimited).

If the customers arrival depends on the number of customers already in the system (in service plus in queue), the calling population is called limited or finite.

Example:

A factory having only four machines which often require repair/service and two of them (say) are in working condition. Then at any point in time, there are only two machines which could possibly require service.

Alternatively, if new customer’s arrival is independent of the number of customers already in the system, the calling population is called unlimited or infinite.

Example:

The customers arriving at a bank or super market, students arriving to get admission at the university, cars arriving at a highway petrol station, etc.

Behaviour of arrivals

If a customer, on arriving at the service system wait in the queue until served and do not switch between waiting lines is called a *patient customer*.

Example:

Machines arrived at the maintenance shop

Whereas the customer, who waits for a certain time in the queue and leaves the service system without getting service due to certain reasons is called an impatient customer.

Example:

A customer who has just arrived at a grocery store and finds that the salesmen are busy in serving the customers already in the system, will either wait for service till his patience is exhausted or estimates that his waiting time may be excessive and so leaves immediately to seek service elsewhere.

Key situations in arrivals

Balking: Customers do not join the queue either by seeing the number of customers already in service system or estimating the excessive waiting time for desired service.

Reneging: Customers after joining the queue, wait for sometimes in the queue but leave before being served on account of certain reasons.

Jockeying: Customers move from one queue to another hoping to receive service more quickly (a common scene at railway booking window).

Pattern of arrivals at the system

Customers may arrive in batches (such as the arrival of a family at a restaurant) or individually (such as the arrival of a train at a platform). These customers may arrive at a service facility either on scheduled time (by prior information) or unscheduled time (without information).

The arrival process (or pattern) of customers to the service system is classified into two categories: *static* and *dynamic*.

These two are further classified based on the nature of arrival time/rate and the control which can be exercised on the arrivals.

In *static arrival pattern*, the control depends on the nature of arrival rate (random or constant). In random (or unscheduled) arrivals the times are random variable and therefore we need to understand the average and frequency distribution of the times.

The *dynamic arrival pattern*, controlled by both service facility and customers. The service facility adjusts its capacity to match changes in the service intensity, by either varying the staffing levels at different timings of service, varying service charges (such as telephone call charges at different hours of the day or week) at different timings, or allowing entry with appointments.

The arrival time distribution can be approximated by one of the following probability distributions:

- a. Poisson distribution
- b. Exponential distribution

The *Poisson distribution* is a discrete probability that provides probabilities for the number of customers that may arrive in any specific interval time, whereas *exponential distribution* provides probabilities for times gap between two consecutive arrivals.

Probability functions:

Let n customers arrive during a time interval '0' to 't'. If ' λ ' is the expected (or average) number of arrivals per unit time, then

- a. The Poisson expected number of arrivals will be λ
- b. The exponential expected number of arrivals will be $\frac{1}{\lambda}$

Queuing process or Arrival process

The way in which customers arrive at the service facility individually or in batches at scheduled or unscheduled time is called the "*arrival process*".

The queuing process refers to the number of queues, and their respective lengths. The number of queues may be single, multiple or priority queues depend upon the layout of a service system. The length (or size) of the queue depends upon the operational such as physical space, legal restrictions and attitude of the customers.

In certain cases, a service system is unable to accommodate more than the required number of customers at a time. No further customers are allowed to enter until space becomes available to accommodate new customers. Such types of situations are referred to as finite (or limited) source queue.

Example:

- Cinema halls, restaurants, etc.

On the other hand, if a service system is able to accommodate any number of customers at a time, then it is referred to as infinite (or unlimited) source queue.

Example:

- In a sales department where the customer orders are received, there is no restriction on the number of orders that can come in so that a queue of any size can form.

In many other situations, when arriving customers found long queue(s) in front of a service facility, they often do not enter the service system even though additional waiting space is available. The queue length in such cases depends upon the attitude of the customers.

Example:

- When a motorist finds that there are many vehicles waiting at the petrol station, in most of the cases he does not stop at this station and seeks service elsewhere.

Queue discipline

The way of selecting customers for service according to certain rules known as the "*queue discipline*".

There are number of ways in which customers in the queue are served. Some of these are:

- a. Static queue discipline
- b. Dynamic queue discipline

Static queue discipline

These are based on the individual customer's status in the queue. Few of such disciplines are:

First come first served (FCFS): If the customers are served in the order of their arrival, then this is known as the first-come first served queue discipline.

Example:

- Hospitals, petrol stations.

Last come first served (LCFS): In this discipline the last customer arrived is served first because it reduces the time.

Example:

- Cargo, movers and packers, production process.

Dynamic queue discipline

These are based on the individual customer attributes in the queue. Few of such disciplines are:

Service in Random Order (SIRO): Under this rule customers are selected for service at random irrespective of their arrivals in the service system.

Priority Service (PS): Under this rule customers are grouped in priority classes on the basis of some attributes such as service time or urgency and FCFS rule is used within each class to provide service.

Example:

- The payment of telephone or electricity bills by cheque or cash.

Pre-Emptive Priority or Emergency (PEP): Under this rule, an important customer is allowed to enter into the system even if a customer with lower priority is already in service.

Non- Pre-Emptive Priority (NPEP): In this case an important customer goes ahead in the queue, but service is started immediately on the completion of the current service.

Service Process

The way of providing service to the customers is called "*service process*".

It is to be by one or more service facilities, this may be person (bank teller) or space (airport runway, parking).

Service facility may be without server (self service) or include one (single server) or more servers operating either in series as a team or in parallel (multiple servers).

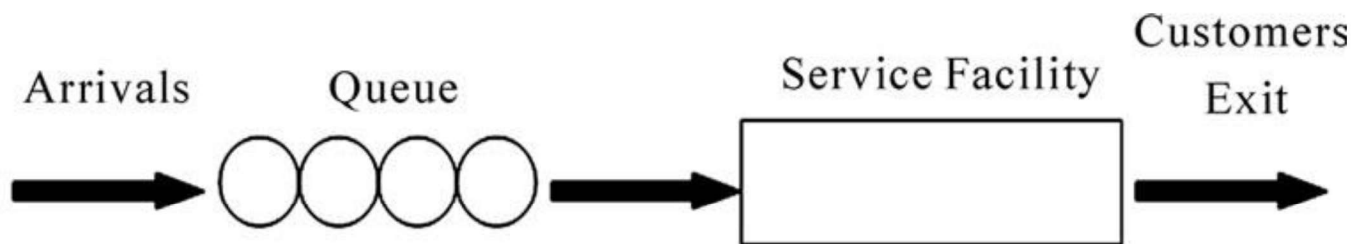
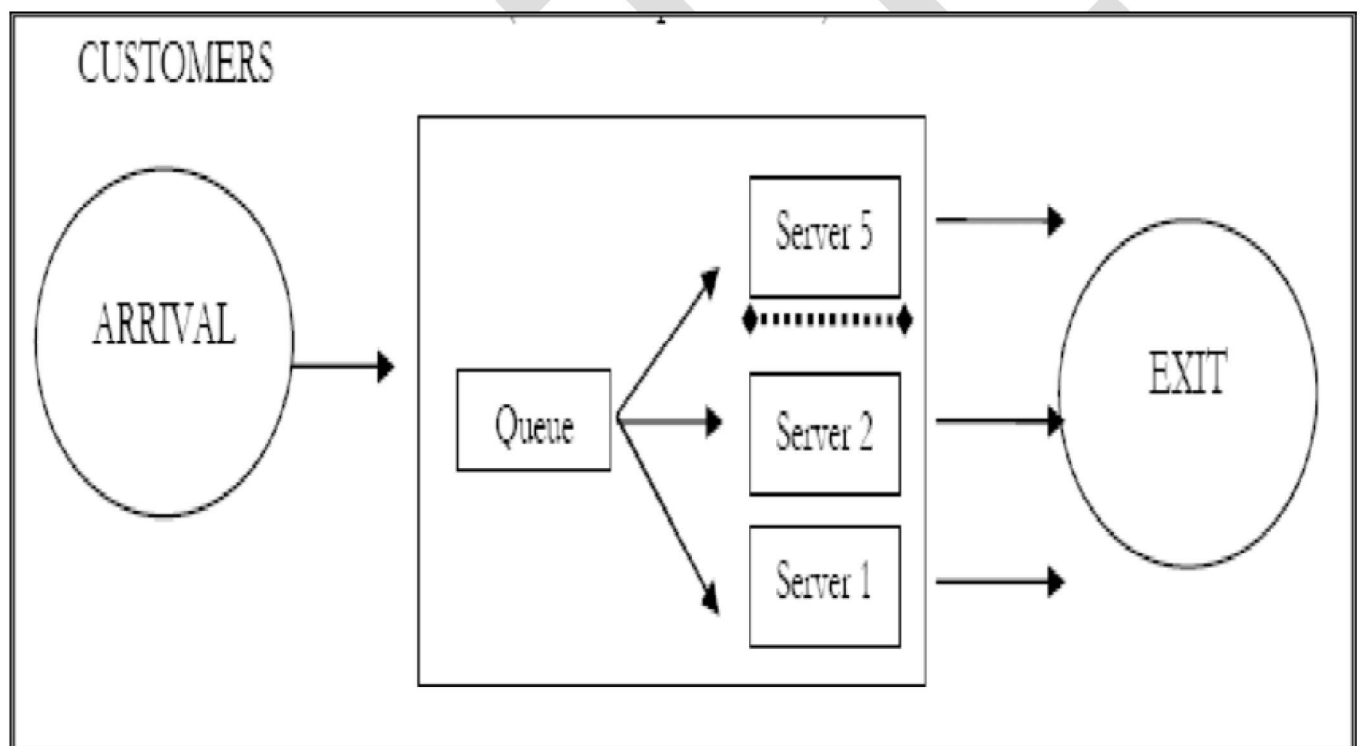
The service process (mechanism) is concerned with the manner in which customers are serviced and leave the service system. It is characterised by two ways:

- a. Arrangement of service facilities

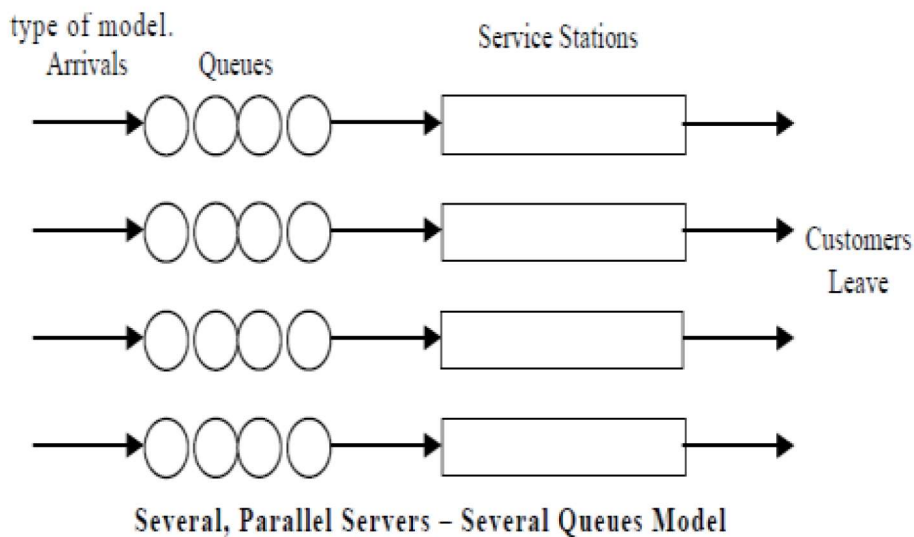
b. Service time distribution

Arrangement of service facilities

The capacity (arrangement) of the service facility is measured in terms of customers that can be served simultaneously and/or effectively. The service facilities (or servers), commonly known as service channels may be in series or in parallel or mixed (partly in series and partly in parallel) as follows:

Single queue, single service facility:***Single queue, multiple service facilities (parallel):***

Multiple queues, multiple service facilities (parallel):



Service time distribution

The time taken by the server from the commencement of service to the completion of service for a customer is known as the service time.

The service time can be approximated by one of the following probability distributions:

- i. Poisson distribution
- ii. Exponential distribution

Let n customers arrive during a time interval '0' to 't'. If ' μ ' is the expected (or average) service rate per unit time, then

- a. The Poisson expected number of arrivals will be μ
- b. The exponential expected number of arrivals will be $\frac{1}{\mu}$

Notations in Queuing Theory

The notations used in the analysis of a queuing system are as follows:

1. n = number of customers in the system (waiting and in service)
2. N = maximum number of customers allowed in the system
3. s = number of service channels (service facilities of servers)
4. q = waiting line or queue
5. λ = average (or expected) customer arrival rate per unit of time.
6. μ = average (or expected) customer service rate per unit of time.
7. ρ = traffic intensity (or) service utilization factor (or) average (or expected) fraction of time for which server is busy.

$$= \frac{\text{average service completion time} \left[\frac{1}{\mu} \right]}{\text{average interarrival time} \left[\frac{1}{\lambda} \right]} = \frac{\lambda}{\mu}$$

8. P_0 = probability of no customer in the system = $(1 - \rho)$
9. P_n = probability of 'n' customer in the system
10. L_s = average (or expected) number of customers in the system
11. L_q = average (or expected) number of customers in the queue
12. W_s = average (or expected) waiting time in the system
13. W_q = average (or expected) waiting time in the queue

SOLUTION OF QUEUING MODELS

Two approaches are available for solving queuing models viz., mathematical and simulation.

In mathematical approach the actual arrival and service time distributions are approximated by one of the known mathematical distributions, i.e., Poisson, exponential, etc., to describe expected value of various operational characteristics of the queuing process.

In this section, we will discuss various queuing models in which both arrivals and departures (those served) take place simultaneously with specific assumptions about the features of queuing systems such as: input source, arrival process, queue configuration, queue discipline and service process.

The simultaneous occurrence of arrivals and departures is also called *birth and death process*. This process helps in determining the probability distribution of the number of customers in the queuing system at a particular instance of time. The probability distribution so obtained is then used to determine performance measures (operating characteristics) of various types of queuing models.

Types of models

1. Single-Server Queuing Models
 - a. Model-I: $\{(M/M/1): (\infty/FCFS)\}$ - Unlimited Queue
 - b. Model-II: $\{(M/M/1): (N/FCFS)\}$ - Limited Queue
2. Multiple-Server Queuing Models
 - a. Model-III: $\{(M/M/s): (\infty/FCFS)\}$ - Unlimited Queue
 - b. Model-IV: $\{(M/M/s): (N/FCFS)\}$ - Limited Queue

Single-Server Queuing Models

Model-I: $\{(M/M/1): (\infty/FCFS)\}$ - Unlimited Queue

This model is based on certain assumptions about the queuing system:

1. Arrivals are described by Poisson probability distribution and come from an infinite population.
2. Single waiting line and each arrival waits to be served regardless of the length of the queue and no balking or reneging.
3. Queue discipline is "FCFS".
4. Single server and service time follow exponential distribution.
5. The average service rate is more than the average arrival rate.

If the probability of the system being empty (no customer) is denoted by P_0 and written as: $P_0 = 1 - \rho$

then, the probability of n customers in the system is denoted by P_n and written as:

$$P_n = \rho^n P_0$$

Performance measures:

$$\begin{aligned} 1. \quad L_s &= \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda} \\ 2. \quad L_q &= \rho \times L_s = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)} \\ 3. \quad W_s &= \frac{L_s}{\lambda} = \frac{1}{\mu-\lambda} \\ 4. \quad W_q &= \frac{L_q}{\lambda} = W_s \times \rho = \frac{\lambda}{\mu(\mu-\lambda)} \end{aligned}$$

5. Probability that number of customer, 'n' in a system exceeds on given number 'k' as written as:

$$P(n \geq k) = \rho^k$$

Model-II: {(M/M/1): (N/FCFS)}-Limited Queue

In this model no more than N customers can be accommodated at any time in the system due to certain reasons. Thus any customer arriving when the system is already contains N customers does not enter the system and is lost.

Example: A finite queue may be arising due to physical constraint such as:

1. Emergency room in a hospital
2. One man barber shop with certain number of chairs for waiting customers etc.

Assumptions of this model are same as that of Model-I except that the length of the queue be limited.

If the number of customers $n=0,1,2,\dots,N$ (or) $n \geq N$

then,

the probability of the system being empties (no customer) is denoted by P_0 and written as:

$$P_0 = \frac{1-\rho}{1-\rho^{(N+1)}}$$

the probability of n customers in the system is denoted by P_n and written as:

$$P_n = \rho^n P_0 ; \rho \neq 1$$

Performance measures:

$$\begin{aligned} 1. \quad L_s &= \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{(N+1)}}{1-\rho^{(N+1)}} \\ 2. \quad L_q &= L_s - \rho \\ 3. \quad W_s &= \frac{L_s}{\lambda[1-P_N]} \\ 4. \quad W_q &= W_s \frac{1}{\mu} = \frac{L_q}{\lambda[1-P_N]} \end{aligned}$$

Where; $P_N = \rho^N P_0$

Multiple-Server Queuing Models

Model-III: $\{(M/M/s): (\infty/FCFS)\}$ -Unlimited Queue

In this case instead of single server, there are multiple but identical servers in parallel to handle arriving customers.

For this queuing system, it is assumed that arrivals follow a Poisson probability distribution at an average rate of λ customers per unit of time and are served on a first-come, first served basis by any of the servers. The service times are distributed exponentially with an average of μ customers per unit of time. It is further assumed that only one queue is formed.

If the probability of the system being empty (no customer) is denoted by P_0 and written as:

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n + \frac{1}{s!} \left(\frac{\lambda}{\mu} \right)^s \left(\frac{s\mu}{s\mu - \lambda} \right) \right]^{-1}$$

Then; the probability of n customers in the system is denoted by P_n and written as:

$$P_n = \frac{\rho^n P_0}{n!}; \rho = \frac{\lambda}{s\mu}$$

Performance measures:

1. $L_q = \left[\frac{(\lambda\mu) \left(\frac{\lambda}{\mu} \right)^s}{(s-1)! (s\mu - \lambda)^2} \right] P_0 = L_s - \frac{\lambda}{\mu}$
2. $L_s = L_q + \frac{\lambda}{\mu}$
3. $W_q = \frac{L_q}{\lambda}$
4. $W_s = W_q + \frac{1}{\mu}$
5. Probability that all servers are simultaneously busy (utilization factor)

$$P(n \geq k) = \left[\frac{1}{s!} \left(\frac{\lambda}{\mu} \right)^s \left(\frac{s\mu}{s\mu - \lambda} \right) \right] P_0$$

Model-IV: $\{(M/M/s): (N/FCFS)\}$ -Limited Queue

If the probability of the system being empty (no customer) is denoted by P_0 and written as:

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n + \sum_{n=s}^N \left(\frac{1}{s!s^{(n-s)}} \right) \left(\frac{\lambda}{\mu} \right)^n \right]^{-1}$$

Then; the probability of n customers in the system is denoted by P_n and written as:

$$P_n = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] P_0; \rho = \frac{\lambda}{s\mu}$$

Performance measures:

1. $L_q = \left[\frac{(s\rho)^s \rho}{s!(1-\rho)^2} \right] [1 - \rho^{(N-s+1)} - (1-\rho)(N-s+1)\rho^{(N-s)}] P_0$
2. $L_s = L_q + \frac{\lambda}{\mu} [1 - P_N]$

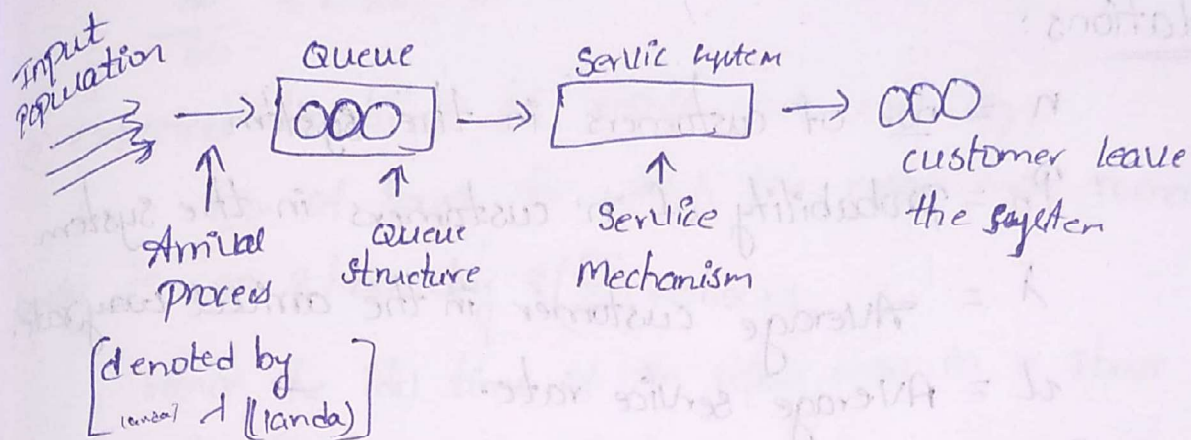
$$3. \quad W_q = \frac{L_q}{\lambda[1-P_N]}$$

$$4. \quad W_s = \frac{L_s}{\lambda[1-P_N]}$$

where; $P_N = P_n$

Unit-V Queuing Theory

Queuing structure:-



Classification of Queuing Models:

Model I: $\{ (M/M/1) : (\infty / FCFs) \}$

M = arrival's distribution

M = service time distribution

1 = No. of service channels (server)

∞ = maximum No. of customers allowed in the queue

FCFS = first come first serve.

Formulas:-

L_s = Expected No. of customers in the system

$$L_s = \frac{\lambda}{\mu - \lambda}$$

$\Rightarrow$ Expected No. of customers waiting in the Queue

$$L_q = \frac{\lambda}{\mu}$$

$\Rightarrow$ waiting time of a customer

$$W_q = \frac{L_q}{\lambda}$$

→ Expected waiting time for a customer in the system. [waiting & service]

$$W_s = \frac{L_s}{\lambda}$$

Notations:

n = no. of customers in the system

P_n = probability of 'n' customers in the system

λ = Average customer in the arrival rate

μ = Average service rate.

$$[\text{rho}]^{\lambda} = \frac{\lambda}{\mu}$$

P_0 = probability of NO customer in system

S = No. of service channels

N = maximum NO. of customers allowed in the system

L_s = Average NO. of customers in the system

L_q = Average NO. of customers in the queue.

W_s = Average waiting time in the system

W_q = Average waiting time in the queue.

Problems:-

A → PD → A

S → ED → U

- 1) A television repairs man finds that the time, spend on his jobs has on exponential distribution with mean of 30 mins. If he repairs the sets follows a Poisson distribution with a approximate rate of 10 per 8 hour day, what is the repairman expected idle time each day? How many jobs are ahead of the average set just brought in?

iii) From the data of the problem we have $\lambda = \frac{100}{84}$

$\Rightarrow \frac{5}{4}$ sets per hour.

$$\mu = \frac{1}{\frac{30}{60}} = 2 \text{ sets per hour}$$

a) expected idle time of repair man each day

No. of hours for which the repair man remains busy in an 8 hour day $8\left(\frac{5}{2}\right) = 5 \text{ hours}$.

$\therefore$ Hence the idle time of an repair man in a 8 hour day will be $(8-5) = 3$

b) Expected No. of T.V sets in the system $= L_s = \frac{\lambda}{\mu - \lambda}$

$$L_s = \frac{\frac{5}{4}}{2 - \frac{5}{4}} = \frac{5}{3}$$

2) In a railway marshalling yard, goods trains arrive at a rate 30 trains per day. Assuming that the interarrival time follows an exponential distribution & the service time (the time taken to hump a train) distribution is also exponential with an average of 36 minutes. Calculate:

a) expected queue size (line length)

b) probability that the queue size exceeds 10.

If the input of trains increases to an average of 33 per day, what will be the change in (i) & (ii)?

801 from the data of the problem, we have

$\lambda = \frac{30}{60} \times 24 = 1/48$ trains per minute & $\mu = 1/36$ trains per minute.

→ The traffic intensity then $\rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75$

(a) Expected queue size (line length):

$$L_s = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3 \text{ trains}$$

(b) Probability that the queue size exceeds 10:

$$P(n \geq 10) = \rho^{10} = (0.75)^{10} = 0.06$$

⇒ If the input increase to 33 trains per day, then we have $\lambda = \frac{33}{60} \times 24$

$$= \frac{11}{480} \text{ trains per minute}$$

$$\mu = \frac{1}{36} \text{ trains per minute}$$

The traffic intensity $\rho = \frac{\lambda}{\mu} = \left(\frac{11}{480}\right)(36) = 0.83$

Hence, recalculating the values for (i) & (ii)

$$L_s = \frac{\rho}{1-\rho} = \frac{0.83}{1-0.83} = 5 \text{ trains}$$

$$P(n \geq 10) = \rho^{10} = (0.83)^{10} = 0.2$$

Queuing models:

single server { Model I : $\{(m|m|1) : (\infty | FCFS)\}$

Model II : $\{(m|m|1) : (N | FCFS)\}$

N : No of persons allowed in the queue

multiple server { Model III : $\{(m|m|s) : (\infty | FCFS)\}$

Model IV : $\{(m|m|s) : (N | FCFS)\}$

Model II :- $\{(m|m|1) : (N | FCFS)\}$

The probability that the queue is empty, $P_0 = \frac{1-p}{1-p^{N+1}}$

where $p = \frac{\lambda}{\mu}$

$$P_N = \frac{1-p}{1-p^{N+1}} p^N \Rightarrow \text{for } n=0, 1, 2,$$

$$L_q = \frac{\lambda}{\mu}, \quad L_s = P_0 \sum_{n=0}^{\infty} n p^n \quad w_s = \frac{L_s}{\lambda}, \quad w_q = \frac{L_q}{\lambda}$$

i) In a Railway marshaling yard good train arrives at the Rate 30 trains per day. Assume that the inter arrival time follows an exponential distribution and the service time also to be assume exponential with mean of 36 mins. (ii) calculate 1st questions:

a) The probability that the yard is empty,

b) The average system length assuming that the line capacity of the yard is 9 trains

Given

sol

$$\mu = 36 \text{ minutes}$$

$$\lambda = \frac{30}{60 \times 24}$$

$$\lambda = \frac{1}{48}$$

$$\mu = \frac{1}{36} \text{ Per minute}$$

$$p = \frac{\lambda}{\mu}$$

$$p = \frac{\frac{1}{48}}{\frac{1}{36}}$$

$$p = 0.75$$

(i) The probability that the add is empty

$$p_0 = 1 - p$$

$$1 - p^{N+1}$$

$$N = 9$$

$$p_0 = \frac{1 - 0.75}{1 - (0.75)^{10}}$$

$$p_0 = 0.2649 \text{ approximate}$$

(ii) The average system length

$$L_s = p_0 \sum_{n=0}^{\infty} n p^n$$

$$L_s = 0.2649 \sum_{n=0}^9 n (0.75)^n$$

$$L_s = 0.8$$

model III:

Length of the queue $L_q = P_s \cdot \frac{\rho}{(1-\rho)^2}$

$$2) \text{ where } P_s = \frac{\left[\frac{\lambda}{\mu}\right]^S}{S!} P_0$$

$$\therefore L_s = \frac{\lambda}{\mu} + L_q$$

$$W_q = \frac{L_q}{\lambda}$$

$$W_s = \frac{L_s}{\lambda}$$

$\therefore$ The mean No. of Individuals who actually wait is given by

$$L_{L>0} = \frac{1}{1-\rho}$$

$\rightarrow$ The mean waiting time in a queue for a customer who actually wait is given by

$$W = \frac{1}{S\mu - \lambda}$$

$\rightarrow$ probability $(W>0) = \frac{P_s}{1-\rho}$

$\rightarrow$ probability that there are some one waiting

$$\frac{P_s \cdot \rho}{1-\rho}$$

$\rightarrow$ Avg no. of idle servers = $S - [\text{avg no. of customers served}]$

$\rightarrow$ efficiency of $[M/M/S]$ model = $\frac{\text{Avg no. of customers served}}{\text{Total no. of customers served}}$